

# MODEL SETS WITH POSITIVE ENTROPY IN EUCLIDEAN CUT AND PROJECT SCHEMES

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**ABSTRACT.** We construct model sets arising from cut and project schemes in Euclidean spaces whose associated Delone dynamical systems have positive topological entropy. The construction works both with windows that are proper and with windows that have empty interior. In a probabilistic construction with randomly generated windows, the entropy almost surely turns out to be proportional to the measure of the boundary of the window.

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## 1. INTRODUCTION

In the last decades, aperiodic order – often referred to as the mathematical theory of quasicrystals – has developed into a broad and highly active field of research, see e.g. [BG,KLS15] for recent books dealing with this topic. In this context, the main attention has been given to models with a strong degree of long-range order. In particular, there is nowadays a fairly good understanding of the relations between pure point diffraction – characterising quasicrystals from the physical viewpoint – and purely discrete dynamical spectrum, which has emerged as one of the major tools in the mathematical analysis of long-range aperiodic order.

In this paper, we have a slightly different focus and construct models that may be considered as intermediate between strong long-range order and disorder. More precisely, we introduce a broad family of model sets, produced by cut and project schemes in Euclidean space, whose associated Delone dynamical systems exhibit a high degree of chaoticity, including positive topological entropy. At the same time, they still inherit a certain degree of long-range order, which is built into the underlying cut and project scheme and manifests itself in a non-vanishing discrete part of the dynamical spectrum as well as in minimality. Although we restrict here to study the basic dynamical properties, we hope that the constructed models may be instrumental in understanding the transition from quasicrystalline to amorphous configurations in solid matter. We note several recent works dealing with similar model sets with ‘thick boundary’ of the window, based on a variety of different methods [BHS16, BJL15, HP13, HR14, KR15]. The reader may take that as an indication for the timeliness of the endeavor.

We will discuss more specifically how the present paper relates to other works and contributes to the emerging general theory towards the end of this section after we have introduced the necessary notation. Here, we already note that – to the best of our knowledge – it provides the first examples of model sets with positive entropy based on Euclidean cut and project schemes.

A *cut and project scheme (CPS)* is a triple  $(G, H, \mathcal{L})$  consisting of locally compact abelian groups  $G$  called *direct space* and  $H$  called *internal space* and a discrete co-compact subgroup (*lattice*)  $\mathcal{L} \subseteq G \times H$  such that the canonical projection  $\pi_G : G \times H \rightarrow G$  is one-to-one and the canonical projection  $\pi_H : G \times H \rightarrow H$  has dense image. This framework goes back to Meyer’s influential book [Mey72] and

has later been developed in [Moo97, Moo00, Sch00]. In this paper we will always take  $G = \mathbb{R}^N$  and we will assume  $H$  to be  $\sigma$ -compact (i.e. a countable union of compact sets) and metrizable. Our main application concerns the case  $G = H = \mathbb{R}$ . So, the reader may also well think from the very beginning of  $H$  as just another Euclidean space  $\mathbb{R}^M$  (where  $M \neq N$  is possible).

Given a relatively compact subset  $W \subseteq H$ , which is called a *window* in this context, such a CPS produces a uniformly discrete subset of  $G$  via

$$\wedge(W) = \pi_G(\mathcal{L} \cap (G \times W)).$$

An alternative way to define  $\wedge(W)$  is to introduce the *star-map*. Set  $L := \pi_G(\mathcal{L})$  and  $L^* := \pi_H(\mathcal{L})$ . Then, the star map  $* : L \rightarrow L^*$  is given by  $\ell \mapsto \ell^*$ , where  $\ell^*$  is uniquely defined by  $(\ell, \ell^*) \in \mathcal{L}$  due to the injectivity of  $\pi_{G|\mathcal{L}}$ . Then, we have

$$\wedge(W) = \{\ell \in L \mid \ell^* \in W\}.$$

If  $W$  has non-empty interior, then  $\wedge(W)$  is called a *model set*, in the general case it is called a *weak model set*. We will be concerned with model sets whose window has a further 'smoothness' feature: A window  $W \subseteq H$  is called *proper* (or sometimes *topologically regular*) if

$$\text{cl}(\text{int}(W)) = W.$$

The associated model set will then also be referred to as *proper model set*. Note that any proper window is compact.

A model set is always uniformly discrete and a proper model set is always a Delone set (see the next section for more detailed definitions and discussion of further facts concerning CPS and model sets).

Given a window  $W \subseteq H$  (which will mostly be compact in our considerations below), we can associate a dynamical system to  $\wedge(W)$  by considering the  $\mathbb{R}^N$ -action  $(s, \Lambda) \mapsto \Lambda - s$  on the *hull* of  $\wedge(W)$ . This hull is given as  $\Omega(\wedge(W)) = \text{cl}(\{\wedge(W) - s \mid s \in \mathbb{R}^N\})$ , where the closure is taken in a suitable topology (defined below). The properties of this dynamical system depend crucially on the boundary of the window  $W$ .

If  $W$  is proper and the boundary of  $W$  has Haar measure zero, then the dynamical system  $(\Omega(\wedge(W)), \mathbb{R}^N)$  is (measurably) isomorphic to the Kronecker flow on the torus  $\mathbb{T} = (\mathbb{R}^N \times H)/\mathcal{L}$  defined by  $\omega : \mathbb{R}^N \times \mathbb{T} \rightarrow \mathbb{T}$ ,  $(s, \xi) \mapsto \xi + [s, 0]_{\mathcal{L}}$  and is therefore uniquely ergodic with purely discrete dynamical spectrum [Sch00] and zero topological entropy [BLR07]. This case has attracted most attention in recent years. In fact, it seems fair to say that *regular model sets* i.e. sets of the form  $\wedge(W)$  for proper  $W$  whose boundary has measure zero are the prime examples for quasicrystals. In particular, substantial efforts have been spent over the years to prove pure point diffraction for regular model sets, see e.g. [Hof96, Sch00]. By now this is well understood. Indeed, the approach of [Hof96] via Poisson summation formula has recently been extended to a very general framework in [RS15]. The result of [Sch00] can be seen within the context of the equivalence between purely discrete dynamical spectrum and pure point diffraction proven in this setting in [LMS02] and later generalized in various directions in e.g. [BL04, Gou04, LS03, LM16]. Moreover, there is also an approach via almost periodicity [BM04], see also [Stru05] for a study of almost periodicity in the context of Meyer sets.

Conversely, the case of windows with 'thick boundary', in the sense of positive Haar measure, is not as well understood. A general idea in this context is that thickness of the boundary should imply positive topological entropy and failure of unique ergodicity. In fact, corresponding conjectures have been brought forward by Moody, see [HR14] for discussion, and Schlottmann [Sch00]. These conjectures are supported by prominent examples. Indeed, for the well-known example of visible

lattice points the associated dynamical system is far from being uniquely ergodic and has positive topological entropy [BMP00, HP13]. This system has still pure point diffraction [BMP00] and pure point dynamical spectrum if it is equipped with a natural ergodic measure [HB14]. Existence of such a canonical ergodic measure for general model sets with thick boundary has received attention recently, see [BHS16] for an approach based on a maximal density condition and [KR15] for an rather structural approach. Quite remarkably all these model sets with maximal density still have pure point diffraction and pure point dynamical spectrum with respect to the canonical measure [BHS16]. Note, however, that the eigenfunctions will in general not be continuous anymore. In this context also a general upper bound on topological entropy has been established [HR14]. Given this support for the mentioned conjectures the recent work [BJL15] comes as quite a surprise as it provides examples of proper model sets with thick boundary which are still uniquely ergodic (and minimal) with topological entropy zero. At the same time [BJL15] also provides some examples of proper model sets with minimal dynamical systems of positive entropy lacking unique ergodicity. All examples of [BJL15] are based on Toeplitz systems.

In all examples in the preceding discussion, where the topological entropy was shown to be positive, the internal space  $H$  is not an Euclidean space but has a rather more complicated structure (being a  $p$ -adic space in the case of the visible lattice points and being an odometer in the case of the Toeplitz systems). In the present paper we provide examples of model sets with positive entropy based on Euclidean internal space.

For the sake of simplicity, we will here restrict to CPS in the Euclidean plane, that is,  $G = H = \mathbb{R}$  (although, in principle it should be possible to carry out similar constructions in higher dimensions as well). Then, a lattice with the above properties is of the form  $\mathcal{L} = A(\mathbb{Z}^2)$ , where  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R})$  satisfies  $a/b, c/d \notin \mathbb{Q}$ . We call such  $\mathcal{L}$  an *irrational lattice*. The situation can be summarised in the following diagram.

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R} \times \mathbb{R} & \xrightarrow{\pi_2} & \mathbb{R} \\ \cup & & \cup & & \cup \\ L & \xleftarrow{1-\cdot} & \mathcal{L} = A(\mathbb{Z}^2) & \xrightarrow{\text{dense}} & L^* \end{array}$$

In this setting we construct examples with positive topological entropy (in fact, the maximal entropy possible given the bound in [HR14]) and lack of unique ergodicity. At the same time these examples still are minimal and have a relatively dense set of continuous eigenvalues. So, our examples share positive entropy and lack of unique ergodicity with the examples of [BMP00, HP13] while they differ from these examples by having the additional regularity feature of minimality and a dense set of continuous eigenvalues. On the other hand our examples share minimality and positive entropy with the mentioned examples of [BJL15] but differ from these examples by being based on a Euclidean CPS. Moreover, they have maximal possible entropy whereas there is no explicit control on the value of the entropy in [BJL15].

To us a main achievement of our construction is that it is rather direct and transparent. By this we hope that it can serve as a tool for further investigations as well.

To give a flavor of our results, we will next state one main theorem (an extended version of which is given below in Theorem 5.4). In the theorem, we focus on a probabilistic model with ‘random’ window. Deterministic constructions are given as well, in Section 6 for the case of model sets and in Section 7 for the case of weak model sets, which has started to attract increasing attention, compare discussion above and [HR14, BHS16].

**Theorem 1.1** *Suppose  $\mathcal{L} \subseteq \mathbb{R}^2$  is an irrational lattice and  $C$  is a Cantor set of positive Lebesgue measure in  $[0, 1]$ . Let  $(G_n)_{n \in \mathbb{N}}$  be a numbering of the bounded connected components of  $\mathbb{R} \setminus C$  and  $\Sigma^+ = \{0, 1\}^{\mathbb{N}}$ . Denote by  $\mathbb{P}$  the Bernoulli distribution on  $\Sigma^+$  with equal probability  $1/2$  for each symbol and define*

$$W(\omega) = C \cup \bigcup_{n \in \mathbb{N}: \omega_n = 1} G_n ,$$

where  $\omega \in \Sigma^+$ . Then for  $\mathbb{P}$ -almost every  $\omega \in \Sigma^+$  the set  $W(\omega)$  is proper and the dynamical system  $(\Omega(\lambda(W(\omega) + \vartheta)), \mathbb{R})$  has positive topological entropy for all  $\vartheta \in \mathbb{R}$  and is minimal for  $\vartheta$  from a residual subset  $\Theta \subseteq \mathbb{R}$  (depending on  $\omega$ ).

- Remark 1.2**
- (a) In fact, the topological entropy attains the upper bound provided in [HR14], which is given in terms of the measure of  $\partial W(\omega)$  and the density of the lattice  $\mathcal{L}$ , see Theorem 5.4.
  - (b) The existence of the residual subset  $\Theta \subseteq \mathbb{R}$  such that for all  $\vartheta \in \Theta$  the system  $(\Omega(\lambda(W(\omega) + \vartheta)), \mathbb{R})$  is minimal is a consequence of general (and well-known) theory of model sets and has nothing to do with our (random) setting.
  - (c) Due to the properness of the window our systems fibre over a torus, i.e. allow for a torus as a factor. This has some consequences: For one thing, by abstract results this then implies that the entropy comes from single fibres (see Remark 2.17 below). In fact, our proof directly exhibits fibres carrying the entropy. Also, having this factor implies that our examples have a relatively dense set of continuous eigenvalues, see Remark 2.10.
  - (d) Our results also show that for the set of  $\omega$  of full measure above and any  $\vartheta \in \mathbb{R}$  the dynamical system  $(\Omega(\lambda(W(\omega) + \vartheta)), \mathbb{R})$  is not uniquely ergodic if  $|C| > 1/2$  holds, see Theorem 5.4.

The reason for the positive topological entropy of  $(\Omega(\lambda(W) + \vartheta), \mathbb{R})$  is the existence of a large ‘random component’ in the hull, which may be of intrinsic conceptual interest. We say  $\Omega(\lambda(W))$  *contains an embedded subshift*, if there exists  $S \subseteq \mathbb{R}^N$  of positive asymptotic density and a uniformly discrete  $U \subseteq \mathbb{R}^N$  such that for any subset  $S'$  of  $S$  there exists  $\Gamma \in \Omega(\lambda(W))$  with  $\Gamma \subseteq U$  and

$$S' = \Gamma \cap S.$$

This means that we may think of the elements of  $S$  as positions of points (or atoms) which may be switched on or off completely independently of each other, without leaving the hull (but there is not control on what happens outside of  $S$  at the same time). Details are discussed in the first part of Section 3. Embedded subshifts are closely related to the local structure of the window  $W$  (or its translate  $W + \vartheta$ ) around the points in  $L^*$ . In later parts of Section 3, we also introduce the notion of local independence of  $W$  with respect to subsets of  $L^*$  to establish criteria for the existence of embedded subshifts. Depending on the context, either a topological (Lemma 3.10) or a metric version (Lemma 3.11) of this concept can be applied. A discussion of failure of unique ergodicity in the presence of embedded subshifts is given in Section 4.

These considerations do not depend on the one-dimensionality of the model and we carry them out for the general CPS introduced above.

The proof of Theorem 1.1 is then given in Section 5. In fact, Theorem 5.4 in that section is an extended version of Theorem 1.1 including parts of Remarks 1.2. Section 6 then provides examples of deterministic windows, again in the Euclidean setting  $G = H = \mathbb{R}$ , that equally lead to positive entropy. While this construction is slightly more technical, it demonstrates that the randomness in the definition of

$W(\omega)$  above is not a key ingredient of the procedure. Moreover, this also sets the ground for the construction of weak model sets (whose window has empty interior) with positive entropy, which is carried out in Section 7.

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## 2. PRELIMINARIES

In this section we discuss the necessary background from the theory of point sets and their associated dynamical systems. The material is essentially well-known. For the convenience of the reader we include some proofs.

**2.1 Delone sets.** A set  $\Lambda \subseteq \mathbb{R}^N$  is called *uniformly discrete* if there exists a real number  $r > 0$  such that

$$(1) \quad \|x - y\| \geq r \quad \text{for all } x, y \in \Lambda$$

where  $\|\cdot\|$  denotes the Euclidean norm. The set is called *relatively dense* if there exists a real number  $R > 0$  such that

$$(2) \quad B_R(x) \cap \Lambda \neq \emptyset \text{ for all } x \in \mathbb{R}^N,$$

where  $B_R(x)$  denotes the closed ball of radius  $R$  around  $x$ . We call  $\Lambda$  a *Delone set* if it is uniformly discrete and relatively dense in  $\mathbb{R}^N$ . We say  $p \in \mathbb{R}^N$  is a *period* of  $\Lambda$  if  $\Lambda + p = \Lambda$  and call  $\Lambda$  *aperiodic* if  $p = 0$  is the only period. Given a Delone set  $\Lambda$ , let  $x \in \Lambda$  and  $\varrho > 0$ . Then the pair  $(P(\varrho, x), \varrho)$  with

$$P(\varrho, x) := (\Lambda - x) \cap B_\varrho(0)$$

is called a  $\varrho$ -*patch of  $\Lambda$  in  $x$* . The *set of all patches* is given by

$$\mathcal{P}(\Lambda) = \{(P(\varrho, x), \varrho) \mid x \in \Lambda, \varrho > 0\}.$$

Note that this definition works also for discrete sets which are not Delone. The set  $\Lambda$  has *finite local complexity* (or *(FLC)* for short) if

$$(FLC) \quad \#\{(\Lambda - x) \cap B_\varrho(0) \mid x \in \Lambda\} < \infty$$

for all  $\varrho > 0$ . This assumption is equivalent to various other properties:

**Lemma 2.1 ([Lag98])** *Let  $\Lambda$  be a Delone set. Then the following statements are equivalent:*

- (i)  $\Lambda$  has (FLC);
- (ii)  $\#\{(\Lambda - x) \cap B_{2R}(0) \mid x \in \Lambda\} < \infty$ , where  $R$  is as in (2);
- (iii)  $\Lambda - \Lambda$  is closed and discrete.

If  $\Lambda$  is a Delone set with  $\Lambda - \Lambda$  uniformly discrete, then  $\Lambda$  is called a *Meyer set*. Being a Meyer set is a notably strong property, and in particular implies (FLC) by Lemma 2.1 (iii).

A Delone set  $\Lambda$  is *repetitive* if for all  $(P, \varrho) \in \mathcal{P}(\Lambda)$  the set

$$\{x \in \Lambda \mid P(\varrho, x) = P\}$$

is relatively dense in  $\mathbb{R}^N$ . It has *uniform patch frequencies* (or *(UPF)* for short) if for all patches  $(P, \varrho) \in \mathcal{P}(\Lambda)$  the limit

$$(UPF) \quad \nu(P, x) = \lim_{n \rightarrow \infty} \frac{\#\{y \in (\Lambda - x) \cap B_n(0) \mid P(\varrho, y) = P\}}{\lambda(B_n(0))}$$

exists and the convergence is uniform in  $x \in \mathbb{R}^N$ . Here,  $\lambda$  denotes the  $N$ -dimensional Lebesgue measure.

**2.2 Cut and project schemes and model sets.** In this section we discuss how Meyer sets arise from CPS.

We adopt the notation introduced in the introduction above and consider a CPS  $(G, H, \mathcal{L})$  with  $G = \mathbb{R}^N$  and  $H$  a locally compact abelian group. We will assume that  $H$  is  $\sigma$ -compact and metrizable.<sup>1</sup> As both  $\mathbb{R}^N$  and  $H$  are  $\sigma$ -compact, the lattice  $\mathcal{L}$  must be countable (as it has a compact quotient). The Haar measure of a measurable subset  $W \subseteq H$  will be denoted by  $|W|$ .

Here are the basic properties of sets arising from the CPS.

**Lemma 2.2 ([Sch00])** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS and  $W \subseteq H$ . Then, the following holds:*

- $\lambda(W)$  is uniformly discrete if  $\text{cl}(W)$  is compact.
- $\lambda(W)$  is relatively dense if  $\text{int}(W) \neq \emptyset$ .

*In particular,  $\lambda(W)$  is Meyer and has (FLC) if  $W$  is relatively compact with non-empty interior.*

*Proof.* We only show how the last statement follows from the first two statements: We have

$$\lambda(W) - \lambda(W) = \{x - y \mid x^*, y^* \in W\} \subseteq \{z \in L \mid z^* \in W - W\} = \lambda(\text{cl}(W) - \text{cl}(W)).$$

Since  $\text{cl}(W) - \text{cl}(W)$  is compact,  $\lambda(\text{cl}(W) - \text{cl}(W))$  is uniformly discrete. Thus, also  $\lambda(W) - \lambda(W)$  is uniformly discrete.  $\square$

If  $L^* \cap \partial W = \emptyset$ , the model set is called *generic*. Here is the fundamental result on proper windows and generic model sets. The result is a consequence of Baire category theorem.

**Lemma 2.3 ([Sch00])** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS and  $W \subseteq H$ . If  $\partial W$  has empty interior, then there exists an  $h \in H$  such that  $W + h$  is generic. In particular, whenever  $W$  is proper there exists  $h \in H$  such that  $W + h$  is generic.*

*Proof.* As  $\mathcal{L}$  is countable and  $\partial W$  has empty interior,

$$L^* - \partial W = \bigcup_{l \in L} (l^* - \partial W)$$

can not agree with  $H$  by Baire category theorem. Now, any  $h \in H \setminus (L^* - \partial W)$  will have the desired property.

The last statement follows as for any proper window  $W$ , clearly, its boundary  $\partial W = W \setminus \text{int}(W)$  has empty interior.  $\square$

A model set  $\lambda(W)$  is called *regular* if  $\lambda(\partial W) = 0$ .

**Lemma 2.4 ([Sch00, Theorem 7.2 and Lemma 7.3])** (a) *Let  $\lambda(W)$  be a regular model set. Then it has (UPF).*  
 (b) *Let  $\lambda(W)$  be a generic model set. Then it is repetitive.*

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<sup>1</sup>Metrizability of  $H$  is only a matter of convenience. It allows us to work with sequences instead of nets. It is clearly met in our specific examples, where we have  $G = H = \mathbb{R}$ .

**2.3 Delone Dynamical Systems.** In this section we show how a uniformly discrete set gives rise to a dynamical system. The dynamical systems arising in this way from Meyer sets are the main object of study in our paper.

Let  $\mathcal{F}$  denote the space of all closed subsets of  $\mathbb{R}^N$  including the empty set. Let furthermore  $\mathcal{U}_r(\mathbb{R}^N)$  be the space of all uniformly discrete sets in  $\mathbb{R}^N$  which satisfy (1) with a fixed constant  $r > 0$ , and  $\mathcal{D}_{r,R}$  be the set of all Delone sets with satisfying (1) and (2) with fixed constants  $r, R > 0$ . We can introduce a metric  $d$  on  $\mathcal{F}$  as follows: Let

$$j: \mathbb{S}^N \longrightarrow \mathbb{R}^N \cup \{\infty\}$$

be the stereographic projection. Here,  $\mathbb{S}^N$  denotes the  $N$ -dimensional sphere in  $\mathbb{R}^{N+1}$  and the point  $\infty$  denotes the additional point in the one-point compactification of  $\mathbb{R}^N$ , which is the image of the ‘north pole’ under  $j$ . Let  $d_H$  be the Hausdorff metric on the set of compact subsets of  $\mathbb{S}^N$ . Then, for any closed  $\Lambda \subseteq \mathbb{R}^N$ , the set  $j^{-1}(\Lambda \cup \{\infty\})$  is a closed and hence compact subset of  $\mathbb{S}^N$ . Thus, via

$$d(\Lambda_1, \Lambda_2) := d_H(j^{-1}(\Lambda_1 \cup \{\infty\}), j^{-1}(\Lambda_2 \cup \{\infty\})),$$

we obtain a topology on the set of all closed subsets of  $\mathbb{R}^N$ .

**Lemma 2.5 ( [LS03] )** *The map  $d: \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}^+$  defines a metric on  $\mathcal{F}$ , which makes  $(\mathcal{F}, d)$  into a compact metric space. Further, the sets  $\mathcal{U}_r$  and  $\mathcal{D}_{r,R}$  are compact in this metric for all  $r, R > 0$ .*

*Proof.* Compactness of  $(\mathcal{F}, d)$  is discussed in [LS03]. As  $\mathcal{U}_r$  and  $\mathcal{D}_{r,R}$  are clearly closed, they are also compact.  $\square$

**Remark 2.6** In the investigation of Delone sets (rather than uniformly discrete sets) another metric may be even more common, see e.g. [LMS02]. However, both metrics induce the same topology, [BL04, LS03].

Let  $\Lambda \subseteq \mathbb{R}^N$  be a uniformly discrete set. Then

$$\Omega(\Lambda) = \text{cl}(\{\Lambda - s \mid s \in \mathbb{R}^N\})$$

is called the *dynamical hull* of  $\Lambda$ . Here, the closure is taken with respect to the topology induced by the metric discussed in Lemma 2.5. Note that this closure may contain the empty set even if  $\Lambda$  was not the empty set. Given the canonical flow  $\varphi_s(\Gamma) := \Gamma - s$  on  $\Omega(\Lambda)$ , we call the pair  $(\Omega(\Lambda), \varphi)$  *point set dynamical system* and also write  $(\Omega(\Lambda), \mathbb{R}^N)$ . Dynamical systems of this form are sometimes called *mathematical quasicrystals*.

**Lemma 2.7 ( [Sch00, Corollary 3.3 and Proposition 3.1] )** *Let  $\Lambda$  be a Delone set with FLC. Then*

- (a)  $(\Omega(\Lambda), \varphi)$  is uniquely ergodic if and only if  $\Lambda$  has (UPF);
- (b)  $(\Omega(\Lambda), \varphi)$  is minimal if and only if  $\Lambda$  is repetitive.

Note that (FLC) is always fulfilled for model sets (as we have already seen above in Lemma 2.2).

The statement of the following proposition is known and discussed within proofs in [Sch00, BLM07].

**Proposition 2.8** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS and  $\Lambda$  a Delone set in  $\mathbb{R}^N$  with  $\Lambda \subseteq L$ . Then, for  $\Gamma \in \Omega(\Lambda)$  the following assertions are equivalent:*

- (i)  $\Gamma \subseteq L$ .
- (ii)  $\Gamma$  contains one point of  $L$ .

In this case, there exists a sequence  $(t_n)$  in  $L$  with  $\Lambda + t_n \rightarrow \Gamma$ .

*Proof.* (i) $\implies$  (ii): This is clear.

(ii) $\implies$  (i): Let  $x \in \Gamma \cap L$  be given. Consider a sequence  $(t_n)$  in  $\mathbb{R}^N$  with  $\Gamma_n := \Lambda + t_n \rightarrow \Gamma$ . Without loss of generality we can then assume  $x \in \Gamma_n$  for all  $n \in \mathbb{N}$ . We then have  $x \in L$  as well as  $x \in L + t_n$  and this implies  $t_n \in L$  for all  $n \in \mathbb{N}$ . This gives, in particular,  $\Gamma_n \subseteq L$  for all  $n \in \mathbb{N}$ . Consider now an arbitrary point  $y \in \Gamma$ . As  $\Gamma_n \rightarrow \Gamma$  and  $x \in \Gamma_n, \Gamma$ , we infer by finite local complexity that  $y \in \Gamma_n$  for all sufficiently large  $n$ . This then implies  $y \in L$ .

The last statement has been proven along the proof of (ii) $\implies$  (i).  $\square$

**2.4 Flow morphism and torus parametrisation.** The dynamical hull of a Delone set arising from a CPS can be described via the so-called *torus parametrization*. This is discussed in this section.

Consider the CPS  $(\mathbb{R}^N, H, \mathcal{L})$  and define the associated *torus* by

$$\mathbb{T} := (\mathbb{R}^N \times H) / \mathcal{L}.$$

Then  $\mathbb{T}$  inherits a natural group structure from  $\mathbb{R}^N \times H$ . We will write  $[s, h]_{\mathcal{L}}$  for the element  $(s, h) + \mathcal{L} \in \mathbb{T}$ .

Then, there is a natural  $\mathbb{R}^N$ -action on  $\mathbb{T}$  given by

$$\omega_s(\xi) := \xi + [s, 0]_{\mathcal{L}}.$$

For  $s \in \mathbb{R}^N$  and  $l \in L$ , we then find

$$\omega_{s-l}(\xi) = \xi + [s - l, 0]_{\mathcal{L}} = \xi + [s, l^*]_{\mathcal{L}}.$$

By the denseness of  $L^*$  in  $H$ , this shows that the action is minimal, i.e. each orbit is dense. As  $\mathbb{T}$  is group, this gives that the action is uniquely ergodic, i.e. there is only one invariant probability measure (see [Sch00] as well).

A *flow morphism* or *factor map* between  $\mathbb{R}^N$ -actions  $(X, \phi)$  and  $(Y, \psi)$  is a continuous onto map  $\eta : X \rightarrow Y$  which satisfies  $\eta(\phi_s(x)) = \psi_s(\eta(x))$  for all  $x \in X$  and  $s \in \mathbb{R}^N$ . If such a flow morphism exists, the dynamical system  $(Y, \psi)$  is called a *factor* of  $(X, \phi)$ .

**Proposition 2.9 ([BLM07])** *Let a CPS  $(\mathbb{R}^N H, \mathcal{L})$ , a proper window  $W \subseteq H$  and  $\Lambda \subseteq \mathbb{R}^N$  with  $\lambda(\text{int}(W)) \subseteq \Lambda \subseteq \lambda(W)$  be given. Then there exists a unique flow morphism  $\beta : \Omega(\Lambda) \rightarrow \mathbb{T}$  with  $\beta(\Lambda) = 0$ . This flow morphism satisfies*

$$(3) \quad \beta(\Gamma) = [s, h]_{\mathcal{L}} \iff \lambda(\text{int}(W) + h) - s \subseteq \Gamma \subseteq \lambda(W + h) - s$$

for  $\Gamma \in \Omega(\Lambda)$ .

The map  $\beta$  from the previous proposition is often called a *torus parametrization*.

**Remark 2.10 (Torus parametrization and continuous eigenfunctions)** Existence of a torus parametrization has consequences for existence of continuous eigenfunctions. Indeed, in the situation of the preceding proposition we can define for any  $\gamma$  in the dual group of  $\mathbb{T}$ , i.e. any continuous group homomorphism  $\gamma : \mathbb{T} \rightarrow \{z \in \mathbb{C} : |z| = 1\} =: S^1$ , the function  $f := f_\gamma := \gamma \circ \beta$  on  $\Omega(\Lambda)$ . This function satisfies

$$f(\phi_s(\Gamma)) = \gamma((\beta(\Gamma)) + [s, 0]_{\mathcal{L}}) = \gamma([s, 0])f(\Gamma) = \gamma^*(s)f(\Gamma)$$

for all  $s \in \mathbb{R}^N$  and  $\Gamma \in \Omega(\Lambda)$ , where we have defined  $\gamma^* : \mathbb{R}^N \rightarrow S^1$ , by  $\gamma^*(s) = \gamma([s, 0])$ . Then,  $\gamma^*$  is an element of the dual group of  $\mathbb{R}^N$  and, hence,  $f$  is a continuous eigenfunction. Moreover, the general theory of CPS shows that the set  $\{\gamma^* : \gamma \in \text{dual group of } \mathbb{T}\}$  is relatively dense in  $\mathbb{R}^N$ . So, we have a relatively dense



set of eigenvalues with continuous eigenfunctions. Indeed, within the minimal point set dynamical systems of finite local complexity this is a characteristic feature of systems consisting of Meyer sets, see [KS14].

The structure of fibres of  $\beta$  will be crucial for our further investigation. From the considerations in [BLM07] we obtain the following lemma. As we will need to build on this argument in later sections we include a short proof.

**Lemma 2.11 ([BLM07])** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS and  $W \subseteq H$  a proper window and  $\Lambda \subseteq \mathbb{R}^N$  with  $\lambda(\text{int}(W)) \subseteq \Lambda \subseteq \lambda(W)$  be given. Then the following dichotomy holds:*

- (a) *If  $\emptyset = (\partial W + h) \cap L^*$  then  $[0, h]_{\mathcal{L}}$  has exactly one preimage under  $\beta$ .*
- (b) *If there exist an  $l \in L$  with  $l^* \in \partial W + h$ , then  $\beta^{-1}([0, h]_{\mathcal{L}})$  contains at least two elements  $\Gamma$  and  $\Gamma'$  which satisfy  $l \in \Gamma$  and  $l \notin \Gamma'$ .*

*In particular,  $[0, h]_{\mathcal{L}}$  has exactly one preimage under  $\beta$  if and only if  $W + h$  is generic i.e.  $(\partial W + h) \cap L^* = \emptyset$  holds.*

*Proof.* Clearly,  $(W + h) \cap L^* = (\text{int}(W) + h) \cap L^*$  if and only if  $(\partial W + h) \cap L^* = \emptyset$ .

Consider first the case  $(\text{int}(W) + h) \cap L^* = (W + h) \cap L^*$ . Then,  $[0, h]_{\mathcal{L}}$  has exactly one preimage under  $\beta$  by (3).

Consider now the case  $l^* \in \partial W + h$  for some  $l^* \in L^*$ . Since  $L^*$  is dense in  $H$  and  $W$  is proper, we can find sequences  $h_n = s_n^*$ ,  $n \in \mathbb{N}$ , and  $h'_n = (s'_n)^*$ ,  $n \in \mathbb{N}$ , with

- $\lim_{n \rightarrow \infty} h_n = \lim_{n \rightarrow \infty} h'_n = h$ ,
- $l^* \in \text{int}(W) + h_n$  and  $l^* \notin W + h'_n$  for all  $n \in \mathbb{N}$ .

By going over to subsequences if necessary, we may assume that  $\{\varphi_{-s_n}(\Lambda)\}_{n \in \mathbb{N}}$  and  $\{\varphi_{-s'_n}(\Lambda)\}_{n \in \mathbb{N}}$  converge to  $\Gamma$  and  $\Gamma'$ , respectively. Since

$$\varphi_{-s_n}(\Lambda) = \Lambda + s_n \supset \lambda(\text{int}(W)) + s_n = \lambda(\text{int}(W) + h_n) \ni l,$$

we obtain  $l \in \Gamma$ . In a similar way, we can show that at the same time  $l \notin \Gamma'$ . Hence, we obtain that  $\Gamma \neq \Gamma'$ . As  $\beta$  is a flow morphism, we have

$$\begin{aligned} \beta(\Gamma) &= \beta\left(\lim_{n \rightarrow \infty} \varphi_{-s_n}(\Lambda)\right) = \lim_{n \rightarrow \infty} \beta(\varphi_{-s_n}(\Lambda)) = \lim_{n \rightarrow \infty} \omega_{-s_n}(\beta(\Lambda)) \\ &= \lim_{n \rightarrow \infty} \omega_{-s_n}(0) = \lim_{n \rightarrow \infty} [-s_n, 0]_{\mathcal{L}} = \lim_{n \rightarrow \infty} [0, h_n]_{\mathcal{L}} = [0, h]_{\mathcal{L}}. \end{aligned}$$

The same holds for  $\Gamma'$ , and hence  $\Gamma, \Gamma' \in \beta^{-1}([0, h]_{\mathcal{L}})$ .

The last statement is immediate from the preceding two statements.  $\square$

From the previous lemma and Lemma 2.3 we immediately infer the following corollary.

**Corollary 2.12** *Consider the situation of the previous lemma. Then, there exists an  $h \in H$  such that the fibre  $\beta^{-1}([0, h]_{\mathcal{L}})$  has only one element.*

**Remark 2.13** The corollary gives that for proper  $W$  the dynamical system  $(\Omega(\Lambda), \mathbb{R}^N)$  is almost-automorphic and  $\mathbb{T}$  is its maximal equicontinuous factor. As we do not use this here, we refrain from further discussion but rather refer the reader to [ABKL15] for a recent survey on this topic and [Auj14] for a recent characterization of Meyer dynamical systems which are almost-automorphic.

From Lemma 2.11 we also immediately obtain that regularity of the window  $W$  has strong implications for the fibre structure. To state this more precisely we need the following piece of notation: Two measure-preserving  $\mathbb{R}^N$ -actions  $(X, \phi, \mu)$  and  $(Y, \psi, \nu)$  are *measure-theoretically isomorphic* if there exist full measure sets  $X_0 \subseteq X$

and  $Y_0 \subseteq Y$  and a measurable bijection  $\eta : X_0 \rightarrow Y_0$  such that  $\eta \circ \phi_s(x) = \psi_s \circ \eta(x)$  for all  $x \in X_0$  and  $s \in \mathbb{R}^N$ .

**Corollary 2.14** ([BLM07]) *Consider the situation of the previous lemma and assume  $|\partial W| = 0$ . Then for  $\lambda$ -almost all  $\xi \in \mathbb{T}$  the preimage  $\beta^{-1}(\xi)$  is a singleton. In particular, the flow  $(\Omega(\Lambda), \varphi)$  is uniquely ergodic and measure-theoretically isomorphic to  $(\mathbb{T}, \omega)$ .*

**Remark 2.15** Whenever we consider Delone dynamical systems which arise from proper model sets, the preceding results on the torus parametrization form a basis for our treatment. However, we will also consider the dynamical hull of weak model sets, which are not proper. In this case, we cannot appeal to the previous results. In fact, if  $W$  is compact with  $\text{int}(W) = \emptyset$  the hull of  $\lambda(W)$  must contain the empty set and there can not exist a torus parametrization as the empty set is fixed by the action, whereas no point of the torus is fixed by the action.

In the sequel, we will often deal with proper windows  $W$  and  $\Lambda = \lambda(W)$ . We will then also need to replace the window  $W$  by any of its translates  $W + \vartheta$ ,  $\vartheta \in H$ . In this case, Proposition 2.9 (applied to  $W + \vartheta$  instead of  $W$ ) yields a unique flow morphism

$$(4) \quad \beta_\vartheta : (\Omega(\lambda(W + \vartheta)), \varphi) \longrightarrow (\mathbb{T}, \omega),$$

which sends  $\lambda(W + \vartheta)$  to 0. For  $\vartheta = 0 \in H$  we will still write  $\beta$  instead of  $\beta_0$ .

**2.5 Uniform distribution and asymptotic densities.** Densities of subsets of Euclidean space will play an important role in our considerations. Here, we discuss the necessary tools.

In the following, we use the partial ordering on  $\mathbb{R}^N$  which is given by  $s \leq t \Leftrightarrow s_i \leq t_i$  for all  $i = 1, \dots, N$ . Given  $t \in \mathbb{R}$ , we let  $\bar{t} = (t, \dots, t) \in \mathbb{R}^N$ . Thus

$$C_t := \{s \in \mathbb{R}^N \mid -\bar{t} \leq s \leq \bar{t}\}$$

is a cube of sidelength  $2t$  and volume  $(2t)^N$ .

Whenever  $S$  is a uniformly discrete subset of  $\mathbb{R}^N$  we define its *asymptotic density* by

$$\nu_S := \limsup_{t \rightarrow \infty} \frac{\#S \cap C_t}{\lambda(C_t)},$$

where  $\#A$  denotes the cardinality of  $A$ . If the limsup is actually a limit, we call it the *density* of the set  $S$ . Model sets provide an instance where densities tend to exist rather generally. This is sometimes discussed under the header 'uniform distribution'. In order to state the corresponding result we will need one more piece of notation: Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS. Then a subset  $F$  of  $\mathbb{R}^N \times H$  is called a *fundamental domain* of  $\mathcal{L}$  if it contains exactly one representative of any element in  $(\mathbb{R}^N \times H)/\mathcal{L}$ . As is well-known the volume of a (measurable) fundamental domain does not depend on the choice of the actual fundamental domain. We denote this volume by  $\text{Vol}(\mathcal{L})$ . We can now recall a result from [Moo02], which gives in our setting the following.

**Theorem 2.16 (Uniform distribution for model sets [Moo02])** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS and  $W \subseteq H$  measurable. Then, the following holds:*

(a) *For almost every  $\vartheta \in H$  (with respect to Haar measure on  $H$ ) the density of  $\lambda(W + \vartheta)$  exists and is given by  $\frac{|W|}{\text{Vol}(\mathcal{L})}$ .*

(b) If  $W$  is compact, the inequality

$$\limsup_{t \rightarrow \infty} \frac{\# \lambda(W + \vartheta) \cap C_t}{\lambda(C_t)} \leq \frac{|W|}{\text{Vol}(\mathcal{L})}$$

holds for all  $\vartheta \in H$ .

(c) If  $W$  is open, the inequality

$$\liminf_{t \rightarrow \infty} \frac{\# \lambda(W + \vartheta) \cap C_t}{\lambda(C_t)} \geq \frac{|W|}{\text{Vol}(\mathcal{L})}$$

holds for all  $\vartheta \in H$ .

*Proof.* Part (a) of the Theorem is shown in [Moo02]. Inspecting the proof there one can easily infer part (b) and (c) as well, see also [HR14] for recent results of this type. For the convenience of the reader we sketch a proof. This proof can be seen as a variant of the considerations in [Moo02]: Consider  $\mathbb{T} = (\mathbb{R}^N \times H)/\mathcal{L}$  and let  $\sigma : \mathbb{R}^N \rightarrow [0, \infty)$  be a continuous function with compact support and  $\int_{\mathbb{R}^N} \sigma ds = 1$ . Define the function

$$f : \mathbb{T} \longrightarrow [0, \infty) \quad , \quad f(\xi) := \sum_{(s,h) \in -\xi} \sigma(s) 1_W(h),$$

where  $1_W$  denotes the characteristic function of  $W$ . Then,  $f$  is a measurable bounded function. (Note that the sum has only finitely many non-vanishing terms as both  $\sigma$  and  $1_W$  vanish outside compact sets.)

Define for  $\xi = [s, h]_{\mathcal{L}}$  the set  $\lambda(\xi) := \lambda(W + h) - s$  and note that this is indeed well defined. Then a short computation (compare [Moo02]) shows that

$$\left| \frac{1}{\lambda(C_t)} \int_{C_t} f(\omega_s(\xi)) ds - \frac{\# \lambda(\xi) \cap C_t}{\lambda(C_t)} \right| \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for all  $\xi \in \mathbb{T}$ . Thus, the desired statements (a),(b), (c) will follow from the corresponding statements for the averages

$$a_t(\xi) := \frac{1}{\lambda(C_t)} \int_{C_t} f(\omega_s(\xi)) ds.$$

These statements in turn hold as  $(\mathbb{T}, \mathbb{R}^N)$  is uniquely ergodic:

(a) Birkhoff's ergodic theorem directly implies convergence of the averages  $a_t(\xi)$  for almost every  $\xi \in \mathbb{T}$ . As convergence for  $\xi$  clearly implies convergence for all  $\omega_s(\xi)$ ,  $s \in \mathbb{R}^N$ , the almost sure convergence in  $\xi \in \mathbb{T}$  implies almost-sure convergence in  $\vartheta \in H$ .

(b) If we replace  $1_W$  by a continuous function with compact support, then  $f$  is continuous and we even have uniform convergence in  $\xi \in \mathbb{T}$  by Oxtoby's theorem. Approximating  $1_W$  from above by continuous functions with compact support we obtain the statement (b) uniformly in  $\xi \in \mathbb{T}$  and hence also in  $\vartheta \in H$ .

(c) This follows by replacing the approximation from above in (b) by approximation from below.  $\square$

**2.6 Topological Entropy.** In this section we introduce the background from entropy theory.

Recall that  $C_t$  denotes the cube of sidelength  $t$  around the origin on  $\mathbb{R}^N$ . Given an  $\mathbb{R}^N$ -action  $\phi$  on a compact metric space  $X$  (whose metric we denote by  $d$ ), we say  $x, x' \in X$  are  $(\varepsilon, t)$ -separated if

$$\max_{s \in C_t} d(\phi_s(x), \phi_s(x')) \geq \varepsilon.$$

A subset  $S \subseteq X$  is called  $(\varepsilon, t)$ -separated if its elements are all pairwise  $(\varepsilon, t)$ -separated. By  $N(\varphi, \varepsilon, t)$  we denote the maximal cardinality of an  $(\varepsilon, t)$ -separated set. The *topological entropy* of  $\phi$  is defined as

$$h_{\text{top}}(\phi) := \lim_{\varepsilon \rightarrow 0} h_{\varepsilon}(\phi) = \sup_{\varepsilon > 0} h_{\varepsilon}(\phi),$$

where

$$h_{\varepsilon}(\phi) = \limsup_{t \rightarrow \infty} \frac{1}{\lambda(C_t)} \log N(\phi, \varepsilon, t).$$

We will be particularly interested in the topological entropy of a dynamical system  $(\Omega(\wedge(W)), \mathbb{R}^N)$  arising from a CPS and a proper  $W$ . In this case, there is a torus parametrization

$$(5) \quad \beta : \Omega(\wedge(W)) \longrightarrow \mathbb{T},$$

due to Proposition 2.9. As  $(\mathbb{T}, \mathbb{R}^N)$  is an isometric flow it has entropy zero. This has some consequences for the topological entropy of  $(\Omega(\wedge(W)), \mathbb{R}^N)$ . As it is instructive to our considerations below we discuss next some abstract background in the subsequent two remarks.

**Remark 2.17 (Positive entropy comes from fibres)** If  $(Y, \psi)$  is a factor of  $(X, \phi)$  we can relate the entropy of the two systems. Indeed, we have

$$h_{\text{top}}(\psi) \leq h_{\text{top}}(\phi)$$

(e.g. [KH97]). It is then also possible to obtain an upper bound on  $h_{\text{top}}(\phi)$  by considering the “the topological entropy realised in single fibres”. In order to be more specific, let  $\eta : X \longrightarrow Y$  be the factor map and denote for any  $\xi \in Y$ , the maximal cardinality of an  $(\varepsilon, t)$ -separated subset of the fibre  $\eta^{-1}(\xi)$  by  $N^{\xi}(\phi, \varepsilon, t)$ . Now let

$$h_{\text{top}}^{\xi}(\phi) := \lim_{\varepsilon \rightarrow 0} h_{\varepsilon}^{\xi}(\phi), \quad \text{where } h_{\varepsilon}^{\xi}(\phi) := \limsup_{t \rightarrow \infty} \frac{1}{\lambda(C_t)} \log N^{\xi}(\phi, \varepsilon, t).$$

Then, clearly

$$h_{\text{top}}^{\xi}(\phi) \leq h_{\text{top}}(\phi)$$

for any  $\xi \in Y$ . As shown in [Bow71] we furthermore have the bound

$$h_{\text{top}}(\phi) \leq h_{\text{top}}(\psi) + \sup_{\xi \in Y} h_{\text{top}}^{\xi}(\phi).$$

If  $h_{\text{top}}(\psi) = 0$  the two preceding inequalities give

$$h_{\text{top}}(\phi) = \sup_{\xi \in Y} h_{\text{top}}^{\xi}(\phi).$$

So, in this case the (positive) topological entropy of  $\phi$  must be realised already in single fibres. Now, this is exactly the situation described in (5). In line with the preceding considerations our approach to positive entropy of  $(\Omega(\wedge(W)), \varphi)$  below will be based on showing positive entropy already in the fibres. We will do so by exhibiting what we call embedded subshifts (see below for details).

**Remark 2.18 (Positive entropy implies thick boundary)** We also note that whenever  $(X, \phi)$  is uniquely ergodic and is measure theoretically isomorphic to a factor  $(Y, \psi)$  of zero topological entropy then the topological entropy of  $(X, \phi)$  must vanish as well. The reason is that the metric entropy (which we will not define as we do not need it below) is invariant under measure theoretic isomorphisms. Hence, the metric entropy of  $(X, \phi)$  and  $(Y, \psi)$  must agree. As  $(X, \phi)$  is uniquely ergodic its topological entropy agrees with its metric entropy due to a variational principle, see e.g. [TZ91]. In our situation described in (5) we obtain then from Corollary 2.14 that the topological entropy of  $(\Omega(W), \mathbb{R}^N)$  must vanish whenever the boundary of  $W$

has measure zero (see [BLR07] as well for a more general result covering all Delone dynamical systems of finite local complexity with pure point diffraction). Now, this implies that examples of CPS with positive topological entropy will necessarily have thick boundary and indeed this will feature prominently in our constructions below.

### 3. EMBEDDED SUBSHIFTS AND TOPOLOGICAL INDEPENDENCE

In this section, we define a simple criterion, namely the existence of ‘embedded subshifts’, for positive entropy of the dynamical hull of a uniformly discrete set in  $\mathbb{R}^N$ . For hulls coming from (weak) model sets, we then relate this to the local structure of the window and introduce the concepts of local topological and metric independence. These will be the main tools to prove positivity of entropy in the constructions in the later sections.

Whenever we meet a CPS  $(\mathbb{R}^N, H, \mathcal{L})$  in this and the remaining sections the group  $H$  will be metrizable and  $\sigma$ -compact, compare the discussion on Page 6.

**3.1 Embedded subshifts.** Embedded subshifts are our key concept in providing positive topological entropy.

**Definition 3.1 (Embedded subshift)** Let  $\Lambda$  be a uniformly discrete subset of  $\mathbb{R}^N$ . An *embedded subshift* in  $\Omega(\Lambda)$  is a pair  $(\Xi, S)$  consisting of a closed subset  $\Xi$  of  $\Omega(\Lambda)$  and a subset  $S$  of  $\mathbb{R}^N$  such that the following holds:

- The set  $S$  has positive asymptotic density, i.e.

$$\nu_S = \limsup_{t \rightarrow \infty} \frac{\#S \cap C_t}{\lambda(C_t)} > 0.$$

- The set

$$U := \bigcup_{\Gamma \in \Xi} \Gamma$$

is uniformly discrete.

- For any subset  $S'$  of  $S$  there exists a  $\Gamma \in \Xi$  with

$$\Gamma \cap S = S'.$$

The elements of  $S$  above are called *free points* of the embedded subshift. The set  $U$  is called *grid* of the embedded subshift. The quantity  $\nu_S$  is the *asymptotic density* of the embedded subshift.

If  $(\Xi, S)$  is an embedded subshift in  $\Omega(\Lambda)$  with  $\Xi \subseteq \Omega'$  for some  $\Omega' \subseteq \Omega(\Lambda)$  we say that  $\Omega'$  *contains an embedded subshift*.

**Remark 3.2** Consider an embedded subshift with free points  $S$  and grid  $U$ .

(a) We clearly have  $S \subseteq U$ . The points of  $S$  are free in the sense that we can choose any subset of  $S$  and exactly this will be the subset from  $S$  appearing in some  $\Gamma \in \Xi$ . In later arguments we will not only have to control occurrence of points of  $S$  but also non-occurrence of points of  $S$ . We will need the set  $U$  in order to treat this non-occurrence.

(b) We call an embedded subshift  $(\Xi, S)$  with grid  $U$  *maximal* if  $(\Xi, S \cup \{u\})$  is not an embedded subshift for any  $u \in U \setminus S$ . In this case, we may think of the elements of  $U \setminus S$  as points *forced by the embedded subshift*. It is not hard to see (by an induction procedure) that any embedded subshift can be extended to a maximal one.

- (c) Let  $(\Xi, S)$  be an embedded subshift. Then,  $(\tilde{\Xi}, S)$  with

$$\tilde{\Xi} := \text{cl}(\{\Gamma \in \Xi : \Gamma \cap S \neq \emptyset\})$$

will also be an embedded subshift (with  $\Xi \subseteq \tilde{\Xi}$ ). Indeed, the only possible difference between  $\tilde{\Xi}$  and  $\Xi$  are those elements of  $\Xi$ , which do not contain any element of  $S$ .

(d) Consider a CPS  $(\mathbb{R}^N, H, \mathcal{L})$  and a proper window  $W$  and  $\Lambda = \mathcal{L}(W)$ . Then, for  $\xi = [s, h]_{\mathcal{L}}$ , all elements of  $\beta^{-1}(\xi)$  are contained in the uniformly discrete set  $\mathcal{L}(W + h) - s$  by Proposition 2.9. So, for any subset  $\Xi$  of  $\beta^{-1}(\xi)$  we have uniform discreteness of  $\bigcup_{\Gamma \in \Xi} \Gamma$ . So, the uniform discreteness of the grid is automatically satisfied for a subshift embedded in such a fibre. Also, in this situation if  $(\Xi, S)$  is an embedded subshift in the fibre  $\beta^{-1}(\xi)$ , then  $(\beta^{-1}(\xi), S)$  is an embedded subshift as well. From Proposition 2.9 and Lemma 2.11 we then infer that the grid for this subshift is given by  $\mathcal{L}(W + h) - s$ .

(e) Whenever the pair  $(\Xi, S)$  is an embedded subshift, then so is the translated pair  $(\varphi_s(\Xi), \varphi_s(S))$  for any  $s \in \mathbb{R}^N$ .

(f) We will be mostly interested in embedded subshifts contained in either  $\Omega(\Lambda)$  or in the fibres  $\eta^{-1}(\xi) \subseteq \Omega(\Lambda)$  of some flow morphism  $\eta : \Omega(\Lambda) \rightarrow Y$ .

The following provides a simple characterization for existence of an embedded subshift.

**Proposition 3.3** *Let  $\Lambda$  be a uniformly discrete subset of  $\mathbb{R}^N$ . Then,  $\Omega(\Lambda)$  contains an embedded subshift if and only if there exist  $S \subseteq \mathbb{R}^N$  and a uniformly discrete  $U \subseteq \mathbb{R}^N$  with the following two properties:*

- (1) *The set  $S$  has positive asymptotic density.*
- (2) *For all finite  $F \subseteq S$  and  $a \in \{0, 1\}^F$ , there exists a  $\Gamma \in \Omega(\mathcal{L}(W))$  with  $\Gamma \subseteq U$  and such that for  $s \in F$*

$$s \in \Gamma \iff a_s = 1.$$

*Proof.* If  $\Omega(\Lambda)$  contains an embedded subshift there clearly exist  $S \subseteq \mathbb{R}^N$  and a uniformly discrete  $U \subseteq \mathbb{R}^N$  satisfying (1) and (2). Conversely, if there exist  $S \subseteq \mathbb{R}^N$  and a uniformly discrete  $U \subseteq \mathbb{R}^N$  satisfying (1) and (2) we may define

$$\Xi' := \{\Gamma \in \Omega(\Lambda) : \Gamma \cap S \neq \emptyset \text{ and } \Gamma \subseteq U\}.$$

Now, let  $\Xi$  be the closure of  $\Xi'$ . Then, all elements in  $\Xi$  are contained in  $U$  and a simple compactness argument shows that for any subset  $S'$  of  $S$  there exists a  $\Gamma \in \Xi$  with  $\Gamma \cap S = S'$ . Hence,  $(\Xi, S)$  is an embedded subshift contained in  $\Omega(\Lambda)$ .  $\square$

**Remark 3.4** Let  $\Lambda$  be a (weak) model set coming from a CPS  $(\mathbb{R}^N, H, \mathcal{L})$  such that  $\Omega(\Lambda)$  satisfies the conditions (1) and (2) of the preceding proposition. Let  $(\Xi, S)$  be the embedded subshift constructed in the proof of the preceding proposition i.e.  $\Xi := \text{cl}(\Xi')$  with  $\Xi' := \{\Gamma \in \Omega(\Lambda) : \Gamma \cap S \neq \emptyset \text{ and } \Gamma \subseteq U\}$ , and let  $U' = \bigcup_{\Gamma \in \Xi'} \Gamma$ . Then, we have

$$S \subseteq U \subseteq t + L$$

for any  $t \in S$ . Indeed, shifting  $S$  by  $-t$  for  $t \in S$ , we may assume without loss of generality  $t = 0$  and  $0 \in S$ . By  $0 \in L$ , we infer from Proposition 2.8 that any  $\Gamma \in \Xi$  containing 0 must be contained in  $L$ . Hence, we have that

$$\widehat{U} = \bigcap_{\Gamma \in \Xi: 0 \in \Gamma} \Gamma \subseteq L$$

and since  $\widehat{U} \subseteq U$  is a discrete set, a simple compactness argument shows  $\widehat{U} = U'$ . Hence  $U' \subseteq L$ , and since further any  $s \in S$  is clearly contained in  $\widehat{U}$ , this shows the claimed statement.

The relevance of embedded subshifts comes from the following lemma.

**Lemma 3.5 (Embedded subshift implies positive entropy)** *Let  $\Lambda$  be a uniformly discrete subset of  $\mathbb{R}^N$ . If  $\Omega(\Lambda)$  contains an embedded subshift of asymptotic density  $\nu_S$ , then*

$$h_{\text{top}}(\varphi) \geq \nu_S \cdot \log 2.$$

*Proof.* Let  $S$  be the set of free points and  $U$  the grid of the embedded subshift. Let  $r > 0$  such that different points of  $U$  have distance at least  $r$ . Consider  $\Gamma, \Gamma' \in \Omega(\Lambda)$  with  $\Gamma, \Gamma' \subseteq U$  and  $s \in \Gamma$  and  $s \notin \Gamma'$  for some  $s \in S$ . By uniform discreteness of  $U$  the set  $\Gamma'$  then does not contain a point in the ball around  $s$  with radius  $r$ . This gives

$$d(\varphi_s(\Gamma), \varphi_s(\Gamma')) \geq r.$$

Hence, any pair  $\Gamma, \Gamma' \in \Omega(\Lambda(W))$  which satisfies the above for some  $s \in S \cap C_t$  is  $(r, t)$ -separated.

Consider now an arbitrary  $\nu' < \nu_S$ . Then, there exist arbitrarily large  $t$  with

$$\#S \cap C_t \geq \nu' \cdot \lambda(C_t).$$

Hence, by the considerations at the beginning of the proof, we have

$$N(\varphi, r, t) \geq 2^{\nu' \cdot \lambda(C_t)}.$$

This implies

$$h_r(\varphi) \geq \nu' \cdot \log 2.$$

As  $\nu' < \nu_S$  was arbitrary we infer  $h_r(\varphi) \geq \nu_S \log 2$ . Now, the desired statement follows from  $h_{\text{top}}(\varphi) \geq h_r(\varphi)$ .  $\square$

The following lemma underlines the spirit of the constructions of the next section. Similar arguments can be found e.g. in [BLM07].

**Lemma 3.6** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS and  $W \subseteq H$  be a proper window. For given  $[0, h]_{\mathcal{L}} \in \mathbb{T}$ , the following conditions are equivalent:*

- (i)  $\Gamma \in \beta^{-1}([0, h]_{\mathcal{L}})$ ;
- (ii) *There exists a sequence  $h_j \in L^*$  such that  $\lim_{j \rightarrow \infty} h_j = h$  and*

$$\lim_{j \rightarrow \infty} \Lambda(W + h_j) = \Gamma.$$

*Proof.* (i) $\Rightarrow$ (ii): By  $\beta(\Gamma) = [0, h]_{\mathcal{L}}$  and Proposition 2.9 we have

$$\Gamma \subseteq \Lambda(W - h) \subseteq L.$$

Now, from Proposition 2.8 we obtain a sequence  $s_j \in L$  with  $\Lambda - s_j \rightarrow \Gamma$ . Due to the continuity of the flow morphism  $\beta$ , we then obtain

$$[0, h]_{\mathcal{L}} = \beta(\Gamma) = \lim_{j \rightarrow \infty} \beta(\varphi_{s_j}(\Lambda(W))) = \lim_{j \rightarrow \infty} [0, s_j^*]_{\mathcal{L}}.$$

This easily implies convergence of  $h_j := s_j^*$  to  $h \in H$  for  $j \rightarrow \infty$ .

(ii) $\Rightarrow$ (i): This follows immediately from the continuity of  $\beta$ .  $\square$

**3.2 Independence of sets and existence of embedded subshifts.** In this section we provide a condition for existence of an embedded subshift.

Consider a CPS  $(\mathbb{R}^N, H, \mathcal{L})$  and denote the neutral element of  $H$  by 0. We sometimes write  $0 \in H$  in order to distinguish it from the origin in  $\mathbb{R}^N$ .

Consider a (weak) model set arising from the given CPS via a relatively compact  $W \subseteq H$ . Then the problem of finding an embedded subshift with set of free points  $S \subseteq L$  in the associated dynamical system is actually related to analyzing the local structure of the window  $W$  in some neighborhood of the points  $s^*$  for  $s \in S$ . In order to get a first idea on this issue the following observation may be helpful:

Let  $F \subseteq L$  be a finite set,  $a \in \{0, 1\}^F$  arbitrary and  $\vartheta \in H$  be given. Now, assume that

$$\emptyset \neq \left( \bigcap_{s \in F: a_s = 1} (W - s^*) \setminus \bigcup_{s \in F: a_s = 0} (W - s^*) \right) \cap (L^* - \vartheta).$$

Then, there exists an  $l \in L$  satisfying for any  $s \in F$  that  $(l^* - \vartheta) \in W - s^* \iff a_s = 1$ . This gives for  $s \in F$

$$s \in \lambda(W + \vartheta) - l \iff a_s = 1.$$

Thus, the set  $\Gamma := \lambda(W + \vartheta) - l$  respects the choice of  $F \subseteq L$  given by  $a$ . Our dealings below will build on this observation. However, two additional points will come up:

- We have to simultaneously deal with all finite subsets  $F$  of a subset  $S$  of  $L$ . In order to still provide the uniform discrete subset  $U$  necessary for an embedded subshift, we will need to require that the set  $S^* = \{s^* \mid s \in S\}$  is relatively compact (see Lemmas 3.7 and 3.11).
- We will allow for one overall shift by  $h \in H$ .

Motivated by the preceding considerations we give the following definitions: A finite family  $A_s, s \in F$ , of subsets of  $H$  is *independent with respect to*  $D \subseteq H$  if for all  $a \in \{0, 1\}^F$  we have

$$\emptyset \neq \left( \bigcap_{s \in F: a_s = 1} A_s \setminus \bigcup_{s \in F: a_s = 0} A_s \right) \cap D.$$

An infinite family of sets is called *independent with respect to*  $D$  if the condition above holds for each finite subfamily. We say the window  $W$  is *independent in*  $P \subseteq L^*$  *with respect to*  $D$ , if the family  $W - p, p \in P$ , is locally independent around  $V$  with respect to  $D$ .

The following lemma relates these concepts to the existence of embedded subshifts. The lemma is our main tool to construct embedded subshifts (and hence, by Lemma 3.5 examples with positive topological entropy). In fact, we will apply the lemma in two situations, namely for proper  $W$  and  $W$  with empty interior but of positive measure. These situations will then be studied in the two subsequent sections. The lemma is formulated in a general version that includes two parameters  $\vartheta$  and  $h$ . However, for a first reading it might be helpful to set them both to zero.

**Lemma 3.7 (Basic criterion for embedded subshifts)** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS,  $W \subseteq H$  relatively compact and  $\vartheta, h \in H$ . If  $\lambda(W + h)$  possesses a subset  $S$  of positive asymptotic density such that  $S^* = \{s^* : s \in S\}$  is relatively compact and  $W + h$  is independent in  $S^*$  with respect to  $L^* + (h - \vartheta)$ , then  $\Omega(\lambda(W + \vartheta))$  contains an embedded subshift.*

*Proof.* We will show that the conditions (1) and (2) of Proposition 3.3 for existence of an embedded subshift are met for  $S$  as in the statement of the lemma and

$$U := \lambda(W + h - S^*).$$

Note that  $U$  is indeed uniformly discrete as  $W + h - S^*$  is relatively compact, compare Lemma 2.2.

Condition (1) is met by assumption. To show condition (2) fix a finite subfamily  $F \subseteq S$  and  $a \in \{0, 1\}^F$ . Then, by independence of  $W + h$   $S^*$  with respect to



$L^* + (h - \vartheta)$ , we have

$$\emptyset \neq \left( \bigcap_{s \in F: a_s=1} (W + h - s^*) \setminus \bigcup_{s \in F: a_s=0} (W + h - s^*) \right) \cap (L^* + h - \vartheta) .$$

Thus there exists an

$$(6) \quad \bar{m}^* \in (L^* + h - \vartheta)$$

such that

$$(7) \quad \bar{m}^* \in W + h - s^* \Leftrightarrow a_s = 1$$

for all  $s \in F$ . Further, by the symmetry  $L^* = -L^*$  we have

$$(8) \quad \bar{m}^* = h - \vartheta - m^*$$

for some  $m \in L$ . Combining this with (7) we obtain

$$(9) \quad s \in \lambda(W + \vartheta) + m \quad \text{if and only if} \quad a_s = 1 .$$

Moreover, we have

$$\Gamma := \lambda(W + \vartheta) + m = \lambda(W + \vartheta + m^*) \stackrel{(8)}{=} \lambda(W + h - \bar{m}^*) \subseteq U,$$

where we used that  $\bar{m}^*$  belongs to  $W + h - S^*$  by (7) to obtain the last inclusion. Thus  $\Gamma$  belongs to  $\Omega(\lambda(W + \vartheta))$  with  $\Gamma \subseteq U$ , and due to (9) we have

$$s \in \Gamma \quad \text{if and only if} \quad a_s = 1 .$$

This finishes the proof.  $\square$

A slightly more specific notion of independence is given by the following notion. It will be needed in particular to obtain further information about embedded subshifts in the case of proper model sets.

An infinite family  $(A_s)_{s \in P}$  of subsets of  $H$  is said to be *locally independent in*  $0 \in H$  with respect to  $D$  if

$$0 \in \text{cl} \left( \left( \bigcap_{s \in F: a_s=1} A_s \setminus \bigcup_{s \in F: a_s=0} A_s \right) \cap D \right)$$

for any finite subset  $F \subseteq P$  and any  $a \in \{0, 1\}^F$ . The window  $W$  is said to be *locally independent in*  $P \subseteq L$  with respect to  $D$  if the family  $W - p$ ,  $p \in P$ , is locally independent in  $0 \in H$  with respect to  $D$ . Hence, the window  $W$  is locally independent in  $P$  with respect to  $D$  if and only if

$$(10) \quad 0 \in \text{cl} \left( \left( \bigcap_{s \in F: a_s=1} (W - s^*) \setminus \bigcup_{s \in F: a_s=0} (W - s^*) \right) \cap D \right)$$

for any finite family  $F \subseteq P$  and any  $a \in \{0, 1\}^F$ .

**Corollary 3.8** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS,  $W \subseteq H$  relatively compact and  $\vartheta, h \in H$ . Assume that  $\lambda(W + h)$  possesses a subset  $S$  of positive asymptotic density such that  $W + h$  is locally independent in  $S^* = \{s^* : s \in S\}$  with respect to  $L^* + (h - \vartheta)$ . Let  $\beta_\vartheta$  denote the flow morphism described in (4). Then there is an embedded subshift contained in  $\beta_\vartheta^{-1}([0, h - \vartheta]_{\mathcal{L}})$ .*

*Proof.* This follows by extending the proof of the previous lemma. Here are the details: Fix a finite subfamily  $F \subseteq S$  of and  $a \in \{0, 1\}^F$ . Then, due to (10) we can choose  $\bar{m}^*$  such that it satisfies (6) and (7) and additionally require that  $\bar{m}^*$  is

arbitrarily close to 0. This means that we can find a sequence of  $\bar{m}_j^*$  such that (6) and (7) hold for all  $j \in \mathbb{N}$  and at the same time

$$(11) \quad \lim_{j \rightarrow \infty} \bar{m}_j^* = 0.$$

Further, we can fix a relatively compact neighbourhood  $V$  of 0 and assume without loss of generality that  $\bar{m}_j^* \in V$  for all  $j \in \mathbb{N}$ .

If now  $m_j \in L$  are chosen such that  $\bar{m}_j^* = h - \vartheta - m_j^*$ , analogous to (8), then we obtain

$$(12) \quad s \in \Gamma_j := \lambda(W + \vartheta) + m_{pj} \quad \text{if and only if} \quad a_j = 1.$$

Moreover, we have

$$\Gamma_j = \lambda(W + \vartheta + m_j^*) = \lambda(W + h - \bar{m}_j^*) \subseteq \lambda(W + h - V) =: U_1,$$

where  $U_1$  is discrete since  $W + h - V$  is relatively compact (compare Lemma 2.2).

Now,  $(\Gamma_j)$  is a sequence in the compact space  $\Omega(\lambda(W + \vartheta))$ . Hence, it possesses an accumulation point  $\Gamma \in \Omega(\lambda(W + \vartheta))$ . As  $\Gamma_j$  is a subset of  $U_1$  and  $U_1$  is uniformly discrete, convergence of the  $\Gamma_j \rightarrow \Gamma$  and (12) yield

$$s \in \Gamma \quad \text{if and only if} \quad a_s = 1.$$

As  $W$  is proper,  $\beta_\vartheta$  is continuous. This gives

$$\begin{aligned} \beta_\vartheta(\Gamma) &= \lim_{j \rightarrow \infty} \beta_\vartheta(\lambda(W + \vartheta) + m_j) \\ &= \lim_{j \rightarrow \infty} \beta_\vartheta(\lambda(W + \vartheta + m_j^*)) \\ &= \lim_{j \rightarrow \infty} [0, m_j^*]_{\mathcal{L}} \\ (\text{by (8)}) &= \lim_{j \rightarrow \infty} [0, h - \vartheta - \bar{m}_j^*]_{\mathcal{L}} \\ (\text{by (11)}) &= [0, h - \vartheta]_{\mathcal{L}}. \end{aligned}$$

This shows that the  $\Gamma$  constructed above are all contained in the fibre  $\beta_\vartheta^{-1}([0, h - \vartheta]_{\mathcal{L}})$ . Thus, we obtain an embedded subshift in that fibre.  $\square$

**3.3 Local topological independence and proper  $W$ .** In this section we consider the case that  $W$  is proper. We provide a sufficient condition for applicability of Corollary 3.8. This condition is given in Lemma 3.10. Our application to the random model in Theorem 1.1 will be based on that lemma.

We say a finite family of sets  $A_s$ ,  $s \in F$ , of subsets of  $H$  is *locally topologically independent in*  $0 \in H$  if for all  $a \in \{0, 1\}^F$  we have

$$0 \in \text{cl} \left( \text{int} \left( \bigcap_{s \in F: a_s = 1} A_s \setminus \bigcup_{s \in F: a_s = 0} A_s \right) \right).$$

An infinite family of sets is called *locally topologically independent in* 0 if the condition above holds for each finite subfamily. A window  $W$  is *locally topologically independent in*  $P \subseteq L^*$ , if the family  $W - p$ ,  $p \in P$ , is locally topologically independent in 0.

**Lemma 3.9** *Any family of subsets of  $H$ , which is locally topologically independent in  $0 \in H$ , is locally independent in 0 with respect to any dense  $D \subseteq H$ .*

*Proof.* Consider an arbitrary finite subfamily  $A_s$ ,  $s \in F$ , of the original family and let  $a \in \{0, 1\}^F$  be given. Define

$$A(a) = \bigcap_{s \in F: a_s = 1} A_s \setminus \bigcup_{s: a_s = 0} A_s.$$

By assumption we have  $0 \in \text{cl}(\text{int}(A(a)))$ . Since  $D$  is dense in  $H$ , the intersection  $\text{int}(A(a)) \cap D$  is dense in  $\text{int}(A(a))$ . Thus, we can choose a sequence  $(h_j)_{j \in \mathbb{N}}$  in  $\text{int}(A(a)) \cap D$  such that  $\lim_{j \rightarrow \infty} h_j = 0$ .  $\square$

**Lemma 3.10 (Topological criterion for embedded subshifts)** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS,  $W \subseteq H$  a proper window and  $h \in H$ . Assume that there exists a subset  $S$  of  $\lambda(W + h)$  of positive asymptotic density such that  $W + h$  is locally topologically independent in  $S^* = \{s^* : s \in S\}$ . Then, the fibre  $\beta_\vartheta^{-1}([0, h - \vartheta]_{\mathcal{L}})$  contains an embedded subshift for every  $\vartheta \in \mathbb{R}$ .*

*Proof.* As  $W + h$  is locally topologically independent in  $S^*$  and  $L^* + (h - \vartheta)$  is dense in  $\mathbb{R}^N$  for all  $\vartheta \in H$ , the preceding lemma gives that  $W + h$  is locally independent in  $S^*$  with respect to  $L^* + (h - \vartheta)$  for all  $\vartheta \in \mathbb{R}$ . As  $W$  is proper, we can now apply Corollary 3.8 to obtain that the fibre  $\beta_\vartheta^{-1}([0, h - \vartheta]_{\mathcal{L}})$  contains an embedded subshift.  $\square$

**3.4 Metric independence and general  $W$ .** The aim of this section is to adapt the above concepts for the case of weak model sets, that is, to compact windows with empty interior. In this case, we need to replace open sets by sets of positive measure and invoke uniform distribution in order to prove analogous statements. As a result we will obtain a criterion for embedded subshifts for general relatively compact  $W$  with positive measure. This criterion is given in Lemma 3.11.

We say a finite family  $A_s$ ,  $s \in F$ , of subsets of  $H$  is *metrically* if for all  $a \in \{0, 1\}^F$  we have

$$0 < \left| \left( \bigcap_{s \in F: a_s = 1} A_s \setminus \bigcup_{s \in F: a_s = 0} A_s \right) \right|.$$

An infinite family of sets is called *metrically independent* if the condition above holds for each finite subfamily. Further, we say the window  $W$  is *metrically independent* in  $P \subseteq L^*$ , if the family  $W - p$ ,  $p \in P$ , is locally metrically independent around  $V$ .

**Lemma 3.11 (Metric criterion for embedded subshifts)** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS,  $W \subseteq H$  a relatively compact window and  $h \in H$ . Assume that there exists a subset  $S$  of  $\lambda(W + h)$  of positive asymptotic density such that  $S^* = \{s^* : s \in S\}$  is relatively compact and  $W + h$  is metrically independent in  $S^*$ . Then  $\Omega(\lambda(W + \vartheta))$  contains an embedded subshift for almost every  $\vartheta \in H$ .*

*Proof.* Let  $F$  be a finite subset of  $S$  and let  $a \in \{0, 1\}^F$  be given. Consider the family  $W + h - s^*$ ,  $s \in F$ , and define

$$\mathcal{W}(a) = \bigcap_{s \in F: a_s = 1} (W + h - s^*) \setminus \bigcup_{s \in F: a_s = 0} (W + h - s^*).$$

Since  $W + h$  is metrically independent in  $S^*$ , we have

$$0 < |\mathcal{W}(a)|.$$

By uniform distribution, Theorem 2.16, we thus obtain that the density of

$$\lambda(\mathcal{W}(a) - h + \vartheta)$$

is positive for almost every  $\vartheta \in H$ . By excluding a set of measure zero, we therefore obtain a set  $\Theta(a) \subseteq H$  of full measure such that for every  $\vartheta \in \Theta(a)$  the set  $L^* + h - \vartheta$  intersects  $V \cap \mathcal{W}(a)$ . Intersecting over the countable family of all finite  $F \subseteq S$  and  $a \in \{0, 1\}^F$  we obtain a set  $\Theta \subseteq H$  of full measure such that for each  $\vartheta \in \Theta$  the set  $L^* + h - \vartheta$  intersects  $V \cap \mathcal{W}(a)$  for arbitrary  $F \subseteq S$  and  $a \in \{0, 1\}^F$ . Hence,  $W + h$

is locally independent around  $V$  in  $S^*$  with respect to  $L^* + h - \vartheta$  for each  $\vartheta \in \Theta$ . Given this, Lemma 3.7 implies the assertion.  $\square$

**Remark 3.12** As in the two preceding sections, one can also define a concept of local metric independence. Given  $A \subseteq H$ , we define the essential closure of  $A$  as the set

$$\overline{A}^{\text{ess}} = \{h \in H \mid |A \cap U| > 0 \text{ for any open neighbourhood } U \text{ of } h\}.$$

Then we say a family of sets  $A_s \subseteq H$ ,  $s \in P$  is metrically independent in zero if

$$0 \in \overline{\mathcal{W}(a)}^{\text{ess}}$$

for all sets

$$\mathcal{W}(a) = \bigcap_{s \in F: a_s=1} A_s \setminus \bigcup_{s \in F: a_s=0} A_s$$

with  $F \subseteq P$  finite and  $a \in \{0, 1\}^F$ .

As with topological independence for proper model sets, this notion may be used to obtain some more information on the embedded subshift than is given by Lemma 3.11.<sup>2</sup> However, this is less relevant in the application to weak model sets for which we introduce metric independence, since we have no torus parametrisation in this case. We will therefore discuss this issue in more detail and only briefly come back to it in Remark 7.3.

#### 4. EMBEDDED SUBSHIFTS AND UNIQUE ERGODICITY

In this section we study how existence of a embedded subshifts of sufficiently high density prevents unique ergodicity.

Recall that we have defined the asymptotic density of a subset  $\Gamma$  of  $\mathbb{R}^N$  by

$$\nu_\Gamma := \limsup_{t \rightarrow \infty} \frac{\#\Gamma \cap C_t}{\lambda(C_t)}.$$

Let now  $(\mathbb{R}^N, H, \mathcal{L})$  be a cut and project scheme and  $W \subseteq H$  be relatively compact and  $(\Omega(\lambda(W)), \mathbb{R}^N)$  the associated dynamical system. If this system is uniquely ergodic, then, by Lemma 2.7(a), the density  $\lim_{t \rightarrow \infty} \frac{\#\Gamma \cap C_t}{\lambda(C_t)}$  exists for every  $\Gamma \in \Omega(\lambda(W))$  and is independent of  $\Gamma$  (as this density is just the patch frequency of the patch  $(\{0\}, r/2)$ , where  $r$  is the minimal distance between points in  $\Gamma$ ). Based on this observation we can now show that  $(\Omega(\lambda(W)), \mathbb{R}^N)$  can not be uniquely ergodic if it contains an embedded subshift with set of free points  $S$  and grid  $U$  such that  $\nu_S$  is large compared to  $\nu_U$ .

**Proposition 4.1** *Let  $(\mathbb{R}^N, H, \mathcal{L})$  be a CPS and  $W$  a relatively compact window and suppose that  $\Omega(\lambda(W))$  contains an embedded subshift with set of free points  $S$  and grid  $U$  and  $\nu_S > \nu_U/2$ . Then  $(\Omega(\lambda(W)), \varphi)$  is not uniquely ergodic. This applies in particular if  $W$  is proper and  $\Omega(\lambda(W))$  contains a subshift embedded in a fibre with asymptotic density  $\nu_S > |W|/2$ .*

*Proof.* Let  $(\Xi, S)$  be the embedded subshift in question. Let  $\Gamma_1, \Gamma_0 \in \Xi$  be given with  $\Gamma_1 \cap S = S$  and  $\Gamma_0 \cap S = \emptyset$ . Then

$$\limsup_{t \rightarrow \infty} \frac{\#\Gamma_1 \cap C_t}{\lambda(C_t)} \geq \limsup_{t \rightarrow \infty} \frac{\#S \cap C_t}{\lambda(C_t)} = \nu_S > \frac{\nu_U}{2}$$

---

<sup>2</sup>In contrast to the topological setting, however, the metrisability (or at least first countability) of the group  $H$  is essential in this case.

but at the same time

$$\liminf_{t \rightarrow \infty} \frac{\sharp \Gamma_0 \cap C_t}{\lambda(C_t)} \leq \liminf_{t \rightarrow \infty} \frac{\sharp (U \setminus S) \cap C_t}{\lambda(C_t)} \leq \nu_U - \nu_S < \nu_U/2.$$

This contradicts the existence of uniform patch frequencies discussed above and thus excludes unique ergodicity.

To show the statement for the case of a subshift in a fibre, note that for an embedded subshift in a fibre the grid  $U$  is contained in  $\wedge(W + \vartheta) - s$  for some  $\vartheta \in H$  and  $s \in \mathbb{R}^N$  (compare Remark 3.2 (d)). By uniform distribution, Theorem 2.16 (b), we then have  $\nu_U \leq |W + \vartheta| = |W|$ . Now, the statement follows from the considerations in the first part of the proof.  $\square$

## 5. RANDOM WINDOWS AND POSITIVE ENTROPY

In this section we will provide a proof of the main theorem, Theorem 1.1, presented in the introduction. In fact, we will provide a strengthening of that result. Up to here our discussion involved fairly general CPS. In this section, we will restrict attention to Euclidean planar CPS. More specifically, we will consider the following situation (S):

- $(\mathbb{R}, \mathbb{R}, \mathcal{L})$  is a planar CPS with  $\mathcal{L} = A(\mathbb{Z}^2)$ , where  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R})$  satisfies  $a/b, c/d \notin \mathbb{Q}$ .
- $C \subseteq \mathbb{R}$  is a Cantor set of positive measure in  $[0, 1]$ . Let  $(G_n)_{n \in \mathbb{N}}$  a numbering of the connected components in  $[0, 1] \setminus C$ .

We then define for  $\omega \in \Sigma^+ = \{0, 1\}^{\mathbb{N}}$  the set

$$(13) \quad W(\omega) = C \cup \bigcup_{n: \omega_n=1} G_n.$$

Let  $\mathbb{P}$  be the Bernoulli distribution on  $\Sigma^+$  with probability  $p \in (0, 1)$  (i.e.  $\mathbb{P}$  is the product measure  $\prod_{n \in \mathbb{N}} \mu$ , where  $\mu$  is the measure on  $\{0, 1\}$  which assigns the value  $p$  to  $\{0\}$  and  $1 - p$  to  $\{1\}$ ).

**Lemma 5.1** *For  $\mathbb{P}$ -almost every  $\omega \in \Sigma^+$ , the window  $W(\omega)$  is proper.*

*Proof.* First, the complement of  $W(\omega)$  consists of a union of connected components  $G_n$  of  $C \setminus [0, 1]$  (those with  $\omega_n = 0$ ). Since these are all open,  $W(\omega)$  is compact. Further, as  $\bigcup_{n: \omega_n=1} G_n$  belongs to the interior of  $W(\omega)$ , we have

$$\partial W(\omega) \subseteq C.$$

Next, we are going to show the reverse inclusion (almost surely). Suppose  $x \in C$ . Since  $C$  is perfect, there exists a sequence of gaps  $\{G_k\}_{k \in \mathbb{N}}$  such that  $\inf G_k \rightarrow x$  for  $k \rightarrow \infty$ . By definition of the window, only intervals  $G_k$  with  $a_k(\omega) = 1$  are included in  $W(\omega)$ . Since all random variables are independent, we have

$$\mathbb{P}(\{G_k \text{ is included in } W(\omega) \text{ for infinitely many } k\}) = 1,$$

$$\mathbb{P}(\{G_k \text{ is not included in } W(\omega) \text{ for infinitely many } k\}) = 1.$$

Thus, for  $\mathbb{P}$ -almost every  $\omega$  there exist subsequences  $G_{k_j} \subseteq W(\omega)$  and  $G_{n_l} \subseteq H \setminus W(\omega)$  of  $G_k \subseteq W(\omega)$  such that  $\lim_{j \rightarrow \infty} \inf G_{k_j} = \lim_{l \rightarrow \infty} \inf G_{n_l} = x$ . Hence, we have  $x \in \partial W(\omega)$   $\mathbb{P}$ -almost surely for every fixed  $x \in C$ . Now, let  $M \subseteq C$  be a countable and dense subset of  $C$ . Then for any  $x \in M$  the argument above shows  $x \in \partial W(\omega)$   $\mathbb{P}$ -almost surely. Hence, the countable set  $M$  is contained in  $\partial W(\omega)$   $\mathbb{P}$ -almost surely. Consequently, we also have

$$\text{cl}(M) = C \subseteq \partial W(\omega)$$

$\mathbb{P}$ -almost surely. Together with the other inclusion shown above, this yields

$$\partial(W(\omega)) = C$$

$\mathbb{P}$ -almost surely. From this we obtain

$$\text{int}(W(\omega)) = W(\omega) \setminus \partial(W(\omega)) = \bigcup_{n:\omega_n=1} G_n$$

$\mathbb{P}$ -almost surely. Using this equality and going again through the argument giving  $C \subseteq \partial W(\omega)$ , we then find  $\mathbb{P}$ -almost surely

$$C \subseteq \partial(\text{int}(W(\omega))).$$

From this we then obtain

$$\text{cl}(\text{int}(W(\omega))) = \text{int}(W(\omega)) \cup \partial(\text{int}(W(\omega))) \supset \text{int}(W(\omega)) \cup C = W(\omega)$$

and hence

$$\text{cl}(\text{int}(W(\omega))) = W(\omega)$$

for  $\mathbb{P}$ -almost all  $\omega \in X$ .  $\square$

In the next step, we need to find a suitable  $h \in \mathbb{R}$  and a respective subset  $S$  of  $\wedge(C+h)$  of positive asymptotic density. In order to avoid some technicalities later, it turns out convenient to work with  $\tilde{C} = C \setminus \bigcup_{n \in \mathbb{N}} \partial G_n$ . Note that  $|\tilde{C}| = |C|$ , since the difference  $C \setminus \tilde{C}$  is just the countable set of endpoints of the intervals  $G_n$ .

**Lemma 5.2** *For Lebesgue-almost all  $h \in \mathbb{R}$  the sequence  $\wedge(\tilde{C} + h)$  has asymptotic density given by  $|C|/|\det A|$ .*

*Proof.* This is a direct consequence of uniform distribution, Theorem 2.16. Note that in the case at hand the measure of a fundamental domain is just given by  $|\det A|$ .  $\square$

It remains to prove that the random window  $W(\omega)$  is  $\mathbb{P}$ -almost surely locally topologically independent in the above sequence  $\wedge(W+h)^*$ .

**Lemma 5.3** *Let  $C$  be a Cantor set with positive measure and let  $W(\omega)$  be defined as in (13). Choose  $h \in \mathbb{R}$ . Then for  $\mathbb{P}$ -almost every  $\omega$  the window  $W(\omega) + h$  is locally topologically independent in  $L^* \cap (\tilde{C} + h)$ .*

*Proof.* Without loss of generality we can set  $h = 0$  (as all our later reasoning in this proof will only involve difference of the form  $x - y$  with  $x, y \in W(\omega) + h$ ). Let  $F$  be an arbitrary finite subset of  $L^* \cap \tilde{C}$ . Let

$$\delta_1 = \frac{1}{2} \cdot \min_{x \neq y \in F} |x - y|.$$

Since the union of gaps of a Cantor set is dense in  $[0, 1]$ , there exist gaps  $I_1^x \subseteq (0, \delta_1)$  of  $C - x$ ,  $x \in F$ , such that

$$\bigcap_{x \in F} I_1^x \neq \emptyset.$$

By the choice of  $\delta_1$ , we have  $I_1^x \neq I_1^y$  if  $x \neq y \in F$ . Further, if we let  $\delta_2 = \min\{1, \min_{x \in F} (\inf I_1^x)\}$ , then by the same argument there exist pairwise different  $I_2^x \subseteq (0, \delta_2)$  of  $C - x$  such that

$$\bigcap_{x \in F} I_2^x \neq \emptyset.$$

Proceeding inductively with this construction, in the  $(n+1)$ -st step we define

$$\delta_n = \min \left\{ \frac{1}{n}, \min_{x \in F} (\inf I_n^x) \right\}$$

and choose gaps  $I_{n+1}^x$  of  $C - x$  such that

$$\bigcap_{i \in \mathcal{C}} I_{n+1}^x \neq \emptyset.$$

Now, let  $(G_n)_{n \in \mathbb{N}}$  be a labeling of all gaps of  $C$ . Then by construction, we have  $I_j^x = G_{n_j^x} - x_i$  for some  $n_j^x \in \mathbb{N}$ , and  $n_j^x \neq n_{j'}^x$  if  $(x, j) \neq (x', j')$ . In particular, this means that  $(\omega_{n_j^x})_{j \in \mathbb{N}}^{x \in F}$  is a two-parameter family of identically distributed independent random variables. Therefore, we obtain that for any  $a \in \{0, 1\}^F$  the set

$$\Omega(a) = \{\omega \in \Sigma^+ \mid \exists \text{ infinitely many } j \in \mathbb{N} : \omega_{n_j^x} = 1 \text{ iff } a_x = 1\}$$

has full measure  $\mathbb{P}(\Omega(a)) = 1$ . However, for all  $\omega \in \Omega(a)$ , we have that

$$I_j = \bigcap_{x \in F} I_j^x \subseteq \left( \bigcap_{x \in F: a_x = 1} W(\omega) - x \right) \setminus \left( \bigcup_{x \in F: a_x = 0} W(\omega) - x \right).$$

Since the intervals  $I_j$  are all open and  $\lim_{j \rightarrow \infty} \inf I_j = 0$ , this shows the local topological independence of  $W(\omega)$  in  $F$ . As this works for any finite subfamily  $F$  of  $L^* \cap \tilde{C}$  and there exist only countably many such subfamilies, we obtain local topological independence of  $W(\omega)$  in  $L^* \cap \tilde{C}$  for  $\mathbb{P}$ -almost every  $\omega \in \Sigma^+$ .  $\square$

We can now summarize the preceding considerations in the following theorem.

**Theorem 5.4** *Assume the situation (S) described at the beginning of this section. Then, there exists a subset  $\Sigma_0^+$  of  $\Sigma^+$  of full  $\mathbb{P}$ -measure such that the following holds:*

- (a) *For all  $\omega \in \Sigma_0^+$  and  $\vartheta \in \mathbb{R}$  the Delone dynamical system  $(\Omega(\lambda(W(\omega) + \vartheta)), \mathbb{R})$  contains an embedded subshift in  $\beta_\vartheta^{-1}(\xi)$  for almost every  $\xi \in \mathbb{T}$ .*
- (b) *For all  $\omega \in \Sigma_0^+$  and  $\vartheta \in \mathbb{R}$  the Delone dynamical system  $(\Omega(\lambda(W(\omega) + \vartheta)), \mathbb{R})$  has positive topological entropy  $h_{\text{top}}(\varphi) = \frac{|C| \log 2}{|\det A|}$ .*
- (c) *For every  $\omega \in \Sigma_0^+$  there exists a residual set  $\Theta$  in  $\mathbb{R}$  such that the Delone dynamical system  $(\Omega(\lambda(W(\omega) + \vartheta)), \mathbb{R})$  is minimal for every  $\vartheta \in \Theta$ .*
- (d) *For every  $\omega \in \Sigma_0^+$  and every  $\vartheta \in \mathbb{R}$  the Delone dynamical system  $(\Omega(\lambda(W(\omega) + \vartheta)), \mathbb{R})$  is not uniquely ergodic provided  $C$  additionally satisfies  $|C| > 1/2$ .*

*Proof.* By Lemma 5.1 there exists a set  $\Sigma_1^+$  of full measure in  $\Sigma^+$  such that  $W(\omega)$  is proper for every  $\omega \in \Sigma_1^+$ . By Lemma 5.3 and Fubini's theorem, there exists a set  $\Sigma_2^+$  of full measure in  $\Sigma^+$  such that for every  $\omega \in \Sigma_2^+$  the window  $W(\omega) + h$  is locally topologically independent in  $L^* \cap (\tilde{C} + h)$  for almost every  $h \in \mathbb{R}$ . Set  $\Sigma_0^+ := \Sigma_1^+ \cap \Sigma_2^+$ . Now, consider an arbitrary  $\omega \in \Sigma_0^+$ .

As due to Lemma 5.2 the set  $\lambda(\tilde{C} + h)$  has asymptotic density  $|C|/|\det A|$  for almost every  $h \in \mathbb{R}$ , we obtain that for almost every  $h \in \mathbb{R}$  the assumptions of Lemma 3.10 are satisfied for  $W = W(\omega) + h$  and  $S := \lambda(\tilde{C} + h)$ . Applying Lemma 3.10 we then obtain an embedded subshift in the fibre  $\beta_\vartheta^{-1}(\xi)$  for  $\xi = [0, h - \vartheta]_{\mathcal{L}}$  for almost all  $h \in \mathbb{R}$ . Now, for each such  $h$  we then obviously obtain an embedded subshift as well for  $[t, h - \vartheta]_{\mathcal{L}}$  for all  $t \in \mathbb{R}$ . This gives (a).

As for (b) we note that the proven part (a) together with Lemma 3.5 directly gives  $h_{\text{top}} \geq \frac{|C| \log 2}{|\det A|}$ . On the other hand by the general results of [HR14] we know that  $h_{\text{top}} \leq \frac{|C| \log 2}{\text{Vol}(\mathcal{L})}$ . Combining these inequalities and using  $\text{Vol}(\mathcal{L}) = |\det A|$ , we arrive at the statement (b).

The statement (c) follows from general (and well-known) theory. In fact, it is a direct consequence of Lemma 2.3 combined with (b) of Lemma 2.4 and (b) of Lemma 2.7.

Finally, it remains to show (d). The preceding considerations give almost surely an embedded subshift with set of free points  $S$  satisfying  $\nu_S = \frac{|C|}{\text{Vol}(\mathcal{L})}$ . Clearly, the grid  $U$  must be contained in  $\lambda([0, 1] + h) - t$  for some  $h \in \mathbb{R}$  and  $t \in \mathbb{R}$  and hence satisfies

$$\nu_U \leq \nu_{\lambda([0, 1] + h) - t} \leq \frac{1}{\text{Vol}(\mathcal{L})},$$

where the last inequality follows by uniform distribution (Theorem 2.16). This shows,

$$\nu_S > \frac{\nu_U}{2}$$

and Proposition 4.1 gives the desired statement.  $\square$

**Remark 5.5** Whenever  $W(\omega)$  is proper, the dynamical system  $(\Omega(\lambda(W(\omega) + \vartheta)), \mathbb{R})$  has the torus  $\mathbb{T}$  as its maximal equicontinuous factor and a relatively dense set of continuous eigenvalues for any  $\vartheta \in \mathbb{R}$ , compare Remark 2.10 and Remark 2.13.

## 6. A DETERMINISTIC CONSTRUCTION

In order to prepare for the construction of weak model sets with positive entropy in the next section, we first provide a deterministic construction of proper model sets with positive entropy. To that end, we first define an initial Cantor set  $C_0$  as the starting point of our construction.

**Lemma 6.1** *Let  $l_n^* = nc \pmod{1}$ . Then there exists an increasing sequence  $(n_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$  and a sequence  $\{\varepsilon_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}_{>0}$  such that the open intervals  $I_k = (l_{n_k}^*, l_{n_k}^* + \varepsilon_k) \pmod{1}$  satisfy*

- (i)  $I_j \cap I_k = \emptyset$  for all  $j \neq k$ ,
- (ii)  $\text{cl}(\bigcup_{k \in \mathbb{N}} I_k) = [0, 1]$ ,
- (iii)  $\lim_{k \rightarrow \infty} \frac{k}{n_k} > 0$ .

*Proof.* For simplicity, we work in the additive group  $\mathbb{R}/\mathbb{Z}$  and omit to write  $\pmod{1}$ . Let  $\eta_j = \min_{n=1}^{2^j} d(nc, 0)$  and  $J_j := [jc, jc + \eta_j)$ . Then

$$B := \{n \in \mathbb{N} \mid J_n \cap J_j \neq \emptyset \text{ for some } j < n\}$$

satisfies, for any  $m \in \mathbb{N}$ ,

$$\begin{aligned} \sharp(B \cap [1, m]) &\leq \sum_{j=2}^m \sharp\{n \in \{j, \dots, m\} \mid nc \in B_{\eta_{j-1}}((j-1)c)\} \\ &\leq \sum_{j=2}^m \frac{m-j+1}{2^j} \leq m \sum_{j=2}^{\infty} \frac{1}{2^j} = \frac{m}{2}. \end{aligned}$$

Now let  $n_1 = 0$  and define

$$n_{k+1} = \min\{n > n_k \mid J_n \cap J_{n_j} = \emptyset \text{ for all } j \leq k\}.$$

Then

$$\widehat{B} = \mathbb{N} \setminus \{n_k \mid k \in \mathbb{N}\} \subseteq B.$$

Since we have

$$k = n_k - \sharp(\widehat{B} \cap [1, n_k]) \geq n_k - \sharp(B \cap [1, n_k]) \geq \frac{n_k}{2}$$



for all  $k \in \mathbb{N}$ , property (iii) is satisfied. By defining  $\varepsilon_k = \eta_{n_k}$  and thus  $I_k = J_{n_k}$  properties (i) and (ii) now follow by construction.  $\square$

Note that  $C_0 = [0, 1] \setminus \bigcup_{k \in \mathbb{N}} I_k$  is a Cantor set, since all intervals  $I_k$  are pairwise disjoint and their union is dense in the circle.

**Lemma 6.2** *Let  $C$  be a Cantor set in  $[0, 1]$  such that  $\{0, 1\} \subseteq C$ . Then there exists a sequence of open sets  $A_j \subseteq [0, 1]$  such that*

- (i) *For all  $j \in \mathbb{N}$  the set  $A_j$  is a union of gaps of  $C$ ,*
- (ii)  *$\partial A_j = C$  for all  $j \in \mathbb{N}$ ,*
- (iii) *the family  $(A_j)_{j \in \mathbb{N}}$  is locally topologically independent in 0.*

*Proof.* For any two Cantor sets  $C, C' \subseteq [0, 1]$  with  $\{0, 1\} \subseteq C \cap C'$  exists an orientation-preserving homeomorphism of  $[0, 1]$  which maps  $C$  to  $C'$ . So without loss of generality, we may assume  $C$  that is the middle third Cantor set. Then we can write

$$C = \left\{ \sum_{n=1}^{\infty} 2a_n 3^{-n} \mid a \in \{0, 1\}^{\mathbb{N}} \right\}.$$

Let  $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$  and denote by  $|a|$  the length of  $a \in \mathcal{A}$ . Then

$$(14) \quad G_a = \left( \sum_{n=1}^{|a|} 2a_n 3^{-n} + 3^{-n}, \sum_{n=1}^{|a|} 2a_n 3^{-n} + 2 \cdot 3^{-|a|} \right)$$

are exactly the gaps of  $C$ . We will construct the sets  $A_j$  such that they all contain

$$A = \bigcup_{a \in \mathcal{A}: |a| \in 4\mathbb{N}} G_a$$

but no  $G_a$  with  $a \in 4\mathbb{N} + 1$ . Since all points of  $C$  are approximated by gaps of both types we always have  $\partial A_j = C$ . Thus, properties (i) and (ii) hold.

Let  $a^n = 0^{2n+1}1 \in \{0, 1\}^{2n+2}$ . Choose a countable partition  $(S_j)_{j \in \mathbb{N}}$  of  $\mathbb{N}$  into infinite sets. Further, let  $((M_j, N_j))_{j \in \mathbb{N}}$  be a numbering of all pairs of disjoint finite sets of integers. Then let

$$V_j = \bigcup_{n \in \mathbb{N}: j \in M_n} S_n$$

and

$$A_j = A \cup \bigcup_{l \in V_j} G_{a^l}.$$

For any  $n \in \mathbb{N}$  the set  $S_n$  is a subset of all  $V_j$  with  $j \in M_n$  and disjoint from all  $V_j$  with  $j \in N_n$ . Thus, the set

$$\bigcap_{j \in M_n} A_j \setminus \bigcup_{j \in N_n} A_j$$

contains  $\bigcup_{l \in S_n} G_{a^l}$ . Since  $S_n$  is infinite, this shows the local topological independence required in condition (iii).  $\square$

Now let  $C_0 = [0, 1] \setminus \bigcup_{k \in \mathbb{N}} I_k$  as above and define a window  $W$  by

$$(15) \quad W = C_0 \cup \bigcup_{k \in \mathbb{N}} I_k \cap \text{cl}(A_k + \inf(I_k)).$$

Note that  $\inf(I_k) = l_{n_k}^*$  by construction. Due to

$$W = \text{cl} \left( \bigcup_{k \in \mathbb{N}} I_k \cap W \right) = \text{cl} \left( \bigcup_{k \in \mathbb{N}} I_k \cap \text{cl}(A_k + \inf(I_k)) \right) = \text{cl}(\text{int}(W))$$

the window is proper.

**Theorem 6.3** *Let  $(\mathbb{R}, \mathbb{R}, \mathcal{L})$  be a CPS and  $W$  as in (15). Suppose  $\beta_\vartheta : \Omega(\wedge(W) + \vartheta) \rightarrow \mathbb{T}$  is the corresponding flow morphism from (3) and (4). Further, choose  $S := (l_{n_k})$ , where  $(n_k)_{k \in \mathbb{N}}$  is chosen as in Lemma 6.1. Then the following holds:*

- (a) *For all  $\vartheta \in \mathbb{R}$  the fibre  $\beta^{-1}([0, \vartheta]_{\mathcal{L}})$  contains an embedded subshift. In particular,  $(\Omega(\wedge(W) + \vartheta), \mathbb{R})$  has positive topological entropy for all  $\vartheta \in \mathbb{R}$ .*
- (b) *The system  $(\Omega(\wedge(W) + \vartheta), \mathbb{R})$  is not uniquely ergodic.*

*Proof.* By construction, the local topological independence of  $W$  in  $S^*$  is equivalent to the local topological independence of the sets  $(A_k)_{k \in \mathbb{N}}$  and thus follows from Lemma 6.1(iii). Hence, by Lemma 3.10,  $\beta^{-1}([0, -\vartheta]_{\mathcal{L}})$  contains an embedded subshift, and  $-\vartheta$  can certainly be replaced by  $\vartheta$ . This proves (a).

To prove statement (b), observe that we have  $\nu_S > 1/2$  due to Lemma 6.1. The fact that  $(\Omega(\wedge(W)), \mathbb{R})$  is not uniquely ergodic follows from

$$\nu_S > \frac{1}{2} \geq \frac{|W|}{2},$$

compare Proposition 4.1. □

## 7. WEAK MODEL SETS WITH POSITIVE ENTROPY

In this section, we will modify the construction of the previous section 6 such that the resulting window  $W$  has an empty interior, but the dynamical system  $(\Omega(\wedge(W)), \mathbb{R})$  still has positive topological entropy.

**Lemma 7.1** *Let  $C \subseteq [0, 1]$  be the middle third Cantor set. Then there exists a sequence of sets  $A_j \subseteq [0, 1]$  such that*

- (i)  $C \subseteq \partial A_j$  for all  $j \in \mathbb{N}$ ,
- (ii)  $\text{int}(A_j) = \emptyset$  for all  $j \in \mathbb{N}$ ,
- (iii) the family  $(A_j)_{j \in \mathbb{N}}$  is locally metrically independent in 0.

*Proof.* We can write

$$C = \left\{ \sum_{n=1}^{\infty} 2a_n 3^{-n} \mid a \in \{0, 1\}^{\mathbb{N}} \right\}.$$

As before, let  $\mathcal{A} = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$  and denote by  $|a|$  the length of  $a \in \mathcal{A}$  and by  $G_a$  the gap of  $C$  corresponding to  $a$ . Let  $M$  be another Cantor set in  $[0, 1]$  such that  $\{0, 1\} \subseteq M$ ,  $|M| > 0$  and  $0 \in \text{cl}_{\text{ess}}(M) = \{x \in \mathbb{R} \mid |B_\varepsilon(0) \cap M| > 0 \text{ for all } \varepsilon > 0\}$ . We will construct the sets  $A_j$  such that each set contains  $C$  and, to ensure metric independence, we insert  $M$  into the gaps of  $C$ . Thus, let again  $a^n = 0^{2n+1}1 \in \{0, 1\}^{2n+2}$  and choose a countable partition  $(S_j)_{j \in \mathbb{N}}$  of  $\mathbb{N}$  into infinite sets. Further, let  $(M_j, N_j)_{j \in \mathbb{N}}$  be a numbering of all pairs of disjoint finite sets of integers. Then let

$$V_j := \bigcup_{n \in \mathbb{N}: j \in M_n} S_n$$

and

$$A_j := C \cup \bigcup_{l \in V_j} G_{a^l} \cap (M + \inf(G_{a^l})).$$

Then the first and the second condition follow again by construction. Further, for any  $n \in \mathbb{N}$  the set  $S_n$  is a subset of all  $V_j$  with  $j \in M_n$  and disjoint from all  $V_j$  with  $j \in N_n$ .

Since  $S_n$  is infinite, for any  $\varepsilon > 0$  there exists  $l \in S_n$  such that  $G_{a^l} \subseteq B_\varepsilon(0)$ . Since  $0 \in \text{cl}_{\text{ess}}(M)$ , the set  $G_{a^l} \cap (M + \inf G_{a^l})$  has positive measure. Thus, as

$$G_{a^l} \cap (M + \inf(G_{a^l})) \subseteq B_\varepsilon(0) \cap \left( \bigcap_{j \in M_n} A_j \setminus \bigcup_{j \in N_n} A_j \right),$$

the set on the right has positive measure. Since this holds for all  $\varepsilon > 0$  and the pair  $(M_n, N_n)$  was arbitrary, this shows the metric independence of family  $(A_j)_{j \in \mathbb{N}}$ .  $\square$

Now, let the sequence  $(n_k)_{k \in \mathbb{N}}$  and the intervals  $I_k$  be as in Lemma 6.1. As in the previous section, let  $C_0 = [0, 1] \setminus \bigcup_{k \in \mathbb{N}} I_k$ . Then we define a window  $W$  of empty interior by

$$(16) \quad W = C_0 \cup \bigcup_{k \in \mathbb{N}} I_k \cap (\inf(I_k) + A_k).$$

Note, that we have  $\inf(I_k) = l_{n_k}^*$  by construction of the  $I_k$ .

**Theorem 7.2** *Let  $(\mathbb{R}, \mathbb{R}, \mathcal{L})$  be a CPS and  $W$  as in (16). Further, choose  $S = (l_{n_k})$ , where  $(n_k)_{k \in \mathbb{N}}$  is chosen as in Lemma 6.1. Then the following statements hold:*

- (a) *For almost all  $\vartheta \in \mathbb{R}$  the hull  $\Omega(\lambda(W + \vartheta))$  contains an embedded subshift and  $(\Omega(\lambda(W + \vartheta)), \mathbb{R})$  has positive topological entropy.*
- (b) *The system  $(\Omega(\lambda(W + \vartheta)), \mathbb{R})$  is not uniquely ergodic.*

*Proof.* By construction, the metric independence of  $W$  in  $S^*$  is equivalent to the metric independence of the sets  $(A_k)_{k \in \mathbb{N}}$  and thus follows from Lemma 7.1(iii). Hence, by Lemma 3.11,  $\Omega(\lambda(W + \vartheta))$  contains an embedded subshift for almost all  $\vartheta \in \mathbb{R}$ , which proves (a).

For the second statement, compare the proof of Theorem 6.3(b).  $\square$

**Remark 7.3** By construction, the window  $W$  in (16) has local metric independence in  $S^*$ , in the sense of Remark 3.12. Similar as in Lemma 3.10, this allows to show that the embedded subshift  $\Xi$  which is obtained is contained in  $\lambda(W + \vartheta)$  (that is,  $\Gamma \subseteq \lambda(W + \vartheta)$  for all  $\Gamma \in \Xi$ ). In the case of proper model sets, this was used further to conclude that  $\Xi$  is contained in the fibre  $\beta^{-1}([\vartheta, 0]_{\mathcal{L}})$ . However, for weak model sets there is not analogous statement to that, since a torus parametrisation does not exist in this case.

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